2000

I/1 : $4a^3/3$

2. (a)(i) 6, 1/5 (ii) infinity (iii) zero (b)(i) $2\sin x$ (ii) $x=0:1,\ x\neq 0:0$
   (b)(iii) zero unless $x = n\pi$
3. (a) $\frac{1}{\sqrt{2}}\left(1+x-\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)$ (b) binomial!, $1/(a^2+b^2x^2)^3 = a^{-6}\left(1-\frac{3b^2x^2}{a^2}+\frac{6b^4x^4}{a^4}+\dots\right)$
   (c) $1+\frac{1}{2}x^2+\frac{3}{8}x^4$
4. (a,b,c) or $(a,-b,-c)$ or $(-a,b,-c)$ or $(-a,-b,c)$ $f=3$ Maxima
5. (a) $y=(x^3+x^2+x)\sec x$ (b) $y=(\frac{1}{2}\cos 2x+x^2/2+3/2)^{-1}$
   (c) $y=(1-\cos x)e^{-x^2/2}$ (d) $x=(1-2(y/x)-(y/x)^2)^{-1/2}$
6. (a) (i) A_{ji} (ii) $A_{ik}B_{kj}$ (iii) A_{ii} (iv) $A_{ik}B_{ki}$ (v) $A_{ik}B_{kl}C_{lj}$ (vi) $A_{ik}B_{lk}C_{jl}$ (b) !!!
7. (b)(i) 2 (ii) 10/3 (iii) 5/21 (c) Most probable = 4 = expected
8. (a) – (b) – (c) – (d) $\underline{x} = k(\underline{u}\wedge\underline{v})$
9. (a) $(2,-2,-1)$ (b) $\underline{r}\cdot(\underline{a}\wedge\underline{b}+\underline{b}\wedge\underline{c}+\underline{c}\wedge\underline{a}) = \underline{a}\cdot(\underline{b}\wedge\underline{c})$ (c) $x+2z=4$
10. (a) $(0,0,\pi a^2)$ (b) $\pi a^2\gamma$ (c) $\pi^2a^3/3$ (d) zero and $1+x+xy$
11. (a)(i) $Re=-2$ $Im=0$ (ii) $Re=\ln r$ $Im=\theta+\pi/2+2n\pi$ (b)(i) 0 to $\pi/2$ (ii) $\pi/2$ to π
    (c) (i) $b>0$ (ii) No sol^n (?)
12. $x^2 = \pi/3 + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$ $\quad x = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n}\sin nx$

II/1 : (b) $\mu = 1/u^2$ $\quad -\frac{1}{x+y} + e^{y/(x+y)} = c$

2. –
3. –
4. (a) $(2,-3)$ saddle (b) couldn't be bothered
5. (a) (i) 0 (ii) $(\beta e^{\pi}\cos\beta+\beta-e^{\pi}\sin\beta)/(1+\beta^2)$ (iii) $-\frac{x^2}{4}+\frac{1}{4}$
6. (a)(i) 1 (ii) zero (b) $-z_0^2r^2\pi$ (c) $\nabla\phi = (0, z^2, 2yz)$
7. (d) couldn't be bothered
8. $z = \alpha\sin\pi x\sin\pi y\cos\sqrt{2}\pi\gamma t$; max speed $= \sqrt{2}\pi\gamma\alpha$ at $x=y=1/2$
9. (b) $b_0=1$ $b_1=1$ $b_2=5/2$ $b_3=11/6$ (c) $c_0=c_1=c_2=1$ $c_3=2/3$ (∴ wrong)
10. (b) $\lambda=-3,-6$ or 12 (c) $\frac{1}{\sqrt{3}}(1,-1,1)$, $\frac{1}{\sqrt{2}}(1,0,-1)$, $\frac{1}{\sqrt{6}}(1,2,1)$
11. (a) $y = A_1e^{-2x/\lambda}+A_2e^{x}+e^{-x}/(2\lambda-4)$
    (b) $y = A_1e^{-2x/\lambda}+A_2e^{x}+xe^{x}/(\lambda+2)$
    CF is $y=(Ax+B)e^{x}$; for $\lambda=0$ need one boundary condition
12. (a)(i) continuous, not differentiable (ii) continuous, differentiable
    (b) $1/\lambda^2$.

2001

I/1 (a) $2\hat{i}-\hat{j}-2\hat{k}+\lambda(3\hat{i}+\hat{j}-3\hat{k})$ (b) $\pm\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$ (c) $\frac{1}{2}\sqrt{154}$ (d) $\frac{5}{\sqrt{154}}$

2. $f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^2}\cos(2n-1)x$

3. (a) $\lambda=\cos\theta\pm\sin\theta$ (b) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm1\end{pmatrix}$ (c) $\lambda=1\pm\sin 2\theta$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm1\end{pmatrix}$

4. —

5. (a) touch at $x=1$

6. (a) $\frac{8}{3}abc(a^2+b^2+c^2)$ (b) $4\pi abc$

7. (a)(i) $\begin{pmatrix}2&-10&15\\-6&-3&-6\end{pmatrix}$ (ii) $\begin{pmatrix}6&12\\2&4\end{pmatrix}$ (iii) 48 (iv) $\begin{pmatrix}-74&0&74\\-74&0&74\\-74&0&74\end{pmatrix}$ (b) $\begin{pmatrix}3&-5/2&1/2\\-3&4&-1\\1&-3/2&1/2\end{pmatrix}$

(b) (cont) sol^n is $(0,2,3)$, no sol^n if $\lambda=1$ or 2

8. —

9. (a) $x-x^2/2+x^3/3-x^4/4+\ldots$ (b) $x/a-x^3/2a^3+3x^5/8a^5-\ldots$ (d) $-x-5x^2/2-x^3/3$
(c) $e^{-a^2}(1+2ax+(2a^2-1)x^2+\ldots)$

10. (a)(i) $a_{ij}b_{ji}$ (ii) $a_{ik}b_{kl}c_{lj}$ (iii) n ?! or δ_{ii}; $Tr(ABA^{-1})=Tr(B)$

11. (a)(i) $Re=-1$ $Im=0$ (ii) $Re=0$ $Im=\phi+\frac{\pi}{2}+2n\pi$ (b)(i) $e^{i\theta}$ with $\theta=2n\pi/3$
(b)(ii) $2e^{i\theta}$ with $\theta=2n\pi/3$ (c) $-1\leq b\leq+1$

12. (b) Mean $=0$ Var $=1/6$ (c)(ii) 3

II/1: (a) $y^2=3x^2-11$ (b) $y=(1-\cos 2x)/4\sin x$ (c) $e^{y/x}(1+2ky^2)=2kx^2$

2. nth derivative of $x(x+1)e^{2x}$ is $2^{n-2}e^{2x}(4x^2+4x+4nx+n(n+1))$

3. ?

4. (a) $y=Ae^{-x}+Be^{-2x}+\frac{1}{2}x-\frac{3}{4}-xe^{-x}$ (b) $y=(Ax+B+x^2)e^{-x}+x^2-4x+6$

5. (a)(i) $e+1$ (ii) $e+1$ (b) zero (c) $f=ye^x+y^2z^3$

6. (a) — (b) $h/r=\sqrt{5}-1$ Area $=(\sqrt{5}+1)\pi$

7. (a)(i) 19 (ii) 30 (iii) zero (b) $\lambda=1$ or -6, $\frac{1}{\sqrt{14}}(3,-2,1)$ and $\frac{1}{7}(6,3,2)$ (c) -2

8. (a) $8/3$ (b) $8\pi/3$

9. —

10. $(\partial u/\partial v)_T=(an^2e^{-an/VRT})/V(V-nb)$; $u=TC_v+$ constant

11. (a) $(5/7)^{n-1}(2/7)$ (b)(i) $2/7$ (ii) $2/7$ (iii) $1/21$ (iv) $11/21$ (v) $1/6$

12. (a)(i) Yes: 2 (ii) Yes: 0 (iii) Yes: 1 (b) — (c) Convergent

2002

I/1: (b)(i) $\underline{r}\cdot\underline{b} = \underline{a}\cdot b$ (ii) $\underline{r}\cdot(\underline{a}\wedge\underline{b} + \underline{b}\wedge\underline{c} + \underline{c}\wedge\underline{a}) = \underline{a}\cdot\underline{b}\wedge\underline{c}$

(iii) $\underline{r}\cdot(\underline{b}\wedge\underline{c} - \underline{a}\wedge\underline{c}) = \underline{a}\cdot(\underline{b}\wedge\underline{c})$

2. $x^2 = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$

3. (a) Max at $x = 1/3$; min at $x = 3$

(b)(i) $y = -2x/11 + 24/11$ (ii) $y = 2x - 2$

4. Max 2 at $\pm(0, \sqrt{2}, \sqrt{2})$; Min $\frac{2}{\sqrt{3}}$ at $\pm\left(\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}\right)$

5. (a) $\sqrt{\pi}/2$, $\sqrt{\pi}/4$, $-\frac{1}{2}e^{-x^2} + c$, $-\frac{(x^2+1)}{2}e^{-x^2} + c$ (b) $\frac{1}{2} - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-4}$

6. (a) $A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ (b) $\lambda = 5$ $\mu = 1$

7. (a)(i) $y + 1 = k(x-1)^{1/2}/(x+1)^{1/2}$ (ii) $y = \frac{1}{2}(\sec x + \tan x) + c(\sec x + \tan x)^{-1}$

(b) $y^{-1} = \frac{1}{2}(x^2 + x + 1/2) + \frac{3}{4}e^{2x}$

8. (a) $\mathrm{Tr} = 2$ $\det = 1$ (b) – (c) –

9. F not, G is; all integrals zero except F along (i), which gives 1

10. $f = c(t+b)^{-1/2}$; $\theta(x,t) = \frac{1}{2(t+1/4)^{1/2}}\left[e^{-(x-2)^2/(4t+1)} - e^{-(x+2)^2/(4t+1)}\right]$

11. (a) see below (b) see below (c) $\ln w = \ln\left(\frac{\sin\phi}{1+\cos\phi}\right) + i\pi/2 \pm 2i\pi n$ $n = 0, 1, 2 \ldots$

12. $f(x,y) = \sum_{m=0}^{\infty} \left(8\sinh(2m+1)x \sin(2m+1)y\right)/\left((2m+1)\pi\left(e^{(2m+1)\pi} - e^{-(2m+1)\pi}\right)\right)$

II/1: $\underline{r}\cdot\underline{a} = \underline{c}\cdot\underline{a}/\lambda$ holds, but after that all I can get is a complicated curve

2. (a) $3\sin a^3/a$ (b) $\beta/(\beta^2+\alpha^2)$ (c) $(\beta^2-\alpha^2)/(\beta^2+\alpha^2)$

3. (a) $8\pi\mu a^3/3$ (b) $\lambda a^5/15$

4. $p_r = \binom{n}{r} p^r q^{n-r}$; $\mu = np$ $\sigma = \sqrt{npq}$; Expected correct = 15

Prob is $> 3\sqrt{5}/2\sqrt{45}$ ie $> 1/2$.

5. Unique soln unless $\lambda = 1$ or 2; when $\lambda = 1$ $\lambda = \mu$ and $y = -1$ $z = 1 - \frac{1}{2}\mu$;

when $\lambda = 1$ no solns; when $\lambda = 2$ $y = 1$ $x =$ anything $z = -x/2$ $\mu = 0$.

6. $\underline{\nabla}\cdot\underline{B} = 0$; 2π, 2π, 2π e^{-3x} +

7. (a) $y = A(e^{-2x} - \frac{2}{3}e^{-3x}) + x - \frac{5}{6}$ (b) $y = A\cos\sqrt{2}x + B\sin\sqrt{2}x$

8. (a) not necessarily antisymm and cannot be orthogonal

(b) $\lambda = -4, 3, 8$; $\frac{1}{2}(\sqrt{3}, 0, 1)$, $(0, 1, 0)$, $\frac{1}{2}(1, 0, -\sqrt{3})$

9. (i) $1 + x/2 + x^2/12$ (ii) $1 + 2x + 2x^2$ (iii) $x^2/2 + x^4/12 + x^6/45$

10. (i) all α (ii) $\alpha \neq \pm 1$ (iii) $\alpha > 0$ (iv) all α

11. $\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial s} = 0$

12. saddles at: $(0,0)$, $(\pm 1/2, \pm\sqrt{2})$; Max at $(0, \pm\sqrt{2})$; Min at $(\pm\frac{1}{2}, 0)$

I/11 (a) $z^* = 2e^{-i\phi}$ $z^{-1} = \frac{1}{2}e^{-i\phi}$ $-z = -2e^{i\phi}$ (b) $\cos 2\theta = 2\cos^2\theta - 1$

$\sin 2\theta = 2\cos\theta\sin\theta$

2003

I/1: $(2,1,-1) = -\frac{1}{3}(-1,1,0) + \frac{2}{3}(0,2,1) + \frac{5}{3}(1,0,-1)$

2: (a)(i) 3, 4, 2/3 (ii) e^{-a} (b)(i) 1/3 (ii) $\pi^2/24$

3. (a) $x=0$ saddle; $x=1$ max; $x=-1$ min (c) zero, 2/27

4. $(x+x^3)\ln(1+x^2) - \frac{2x^3}{3} + c$ (i) $\ln(\ln x)+c$ (ii) $\frac{\cos^{-3}x}{3} - \cos^{-1}x + c$

5. (c) $n=3$ except for (iii), where n = anything

6. $u = (3\sin x)e^{-t} - (\sin 3x)e^{-9t}$

7. (i) zero (ii) $\frac{1}{3}(\lambda-\mu-1)$ (iii) $\mu=0$ $\lambda=1$

8. (a)(i) $1+x+x^2/2$ (ii) $x^3 - x^5/2 + 13x^7/120$ (iii) $e + xe + x^2e$

(b) $\frac{1}{a_0} - \frac{a_1}{a_0^2}x + \left(\frac{2a_1^2}{a_0^3} - \frac{2a_2}{a_0^2}\right)\frac{x^2}{2}$

9. (b) $\sinh x = x + x^3/3! + x^5/5! + \ldots$ $\cosh x = 1 + x^2/2! + x^4/4! + \ldots$

(c) $\frac{1}{2}\cosh\frac{\pi}{2}$

10. (b) $5\cdot 2$

11. (a)(i) $y = e^{k\tan^{-1}x + kc}$ (ii) $y = Ae^{x^3/3} - 1$ (b) $\cos\frac{y}{x} = \frac{e^{(1/x-1)}}{\sqrt{2}}$

12. —

II/1: (a) $\underline{r}\cdot\frac{(-1,-2,2)}{3} = \frac{2}{3}$ shortest distance $= \frac{2}{3}$ (b) $\frac{1}{3}(2,-1,-2)$ and $(6,2,5)$

2. $\sin ax = \sum_{n=1}^{\infty} \frac{2n}{\pi}(-1)^n \frac{\sin a\pi \sin nx}{a^2-n^2}$

3. (a) (a,b) is $(0,\pm 2)$ or $(\pm 2, 0)$ (b) $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$ and $\frac{1}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix}$; $\lambda = 1$ or 6.

4. (b) 3/4 (c) $-\pi r^2$

5. (a) $an/(an + b(N-n))$ (b)(i) $A = k$ (ii) e^{-2k} (iii) $1/k$

6. The inverse is $\begin{pmatrix} 2 & -5 & 16 \\ 1 & -3 & 11 \\ -1 & 3 & -10 \end{pmatrix}$

7. grad $= \left(\frac{2x}{a^2}, \frac{2y}{b^2}, 2z\right)$; final integral $= \frac{a^2-b^2}{15}\left[\left(1-\frac{1}{a^2}-\frac{1}{b^2}\right)^{5/2} - \left(1-\frac{1}{b^2}\right)^{5/2} - \left(1-\frac{1}{a^2}\right)^{5/2} + 1\right]$

8. (a)(i) $n-2$ (ii) $\det P = 0$ (b) —

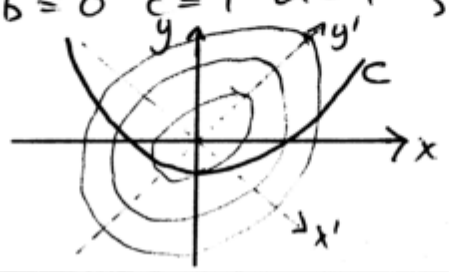
9. (b) $C_p - C_v = Nk$

10. (a) $\frac{\partial^2 r}{\partial x^2} = y^2(x^2+y^2)^{-3/2}$; $\frac{\partial^2 r}{\partial y^2} = x^2(x^2+y^2)^{-3/2}$; $\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial x} = -xy(x^2+y^2)^{-3/2}$

(b) —

11. $a=-1$ $b=0$ $c=1$ $d=1$; $y = (x-1)e^x - \sin x + (1+x)\cos x$

12. wo ho ho



put $x' = x-y$ $y' = x+y$
contours of u follow ellipses
C is as shown. Stationary points occur when C is parallel to contours

2004

I/1. (i) $x+z=4$ (ii) 8

2. (i) conv (ii) div (iii) conv (iv) conv

3 (ii) Both equal $1/2$

4 (a) $h\left\{\frac{(b+1)^2}{(a+1)^2}\frac{a}{b}\right\}$ (b) $\tan x - x + c$ (c) $I_3 < I_1 < I_2$

5. (a) $k=1$ mean $=1$ variance $=1$ (b)(i) $m=1/2$ (ii) mean $=0$ (iii) var $=\frac{3}{2}$ (iv) $1-e^{-1}$

6. (i) $x=1/2$ $y=1/2$ $z=-1$ (ii) $\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix}$ (iii) $\underline{x}=k\begin{pmatrix}-3\\1\\12\end{pmatrix}$ $k=$ any scalar

7. (b)(i) 3 (ii) $1+4(a_1^2+a_2^2+a_3^2)$ (iv) zero

8. (i) 1 and 1 (ii) 1 and 1 (iii) $\Phi = x^3y^2z$ F is not conservative

9. (a)(i) $\cos 2 + i\sin 2$ (ii) $(1+2n)i\pi$ (iii) $(\frac{1}{2}+2n)i\pi$ (iv) $\cos(\ln 10)+i\sin(\ln 10)$
(v) $e^{-(1/2+2n)\pi}$ (c)(i) zero (ii) $\ln 5/4$

10. (a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (b)(i) an ellipse (ii) circle $(3+2i)$, radius 8 (iii) circle $(3i-2)$, radius 8 (c)(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

11. (a) $e^{1/y} = k(1-x^2)^{1/2}$ (b) $y^{-2} = 3e^{-2x}+1$ (c) $y = c\sin x - \sin^2 x - 1$

12. Mass $=\pi\rho h a^2/2$ $d = 7h/30$

II/1. –

2. $3-\sum_{m=1}\frac{12}{(2m-1)\pi}\sin\frac{(2m-1)\pi x}{5}$; converges to 3 in all cases

3. (b)(iii) $E_2 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\sigma\begin{pmatrix}-1&0\\0&1\end{pmatrix}$

4. (a) $4/3$ (b) $4\sin 4\theta/\sin\theta$, 16, $(1,1)$

5. (i) $\alpha = 1-193/256$ so prob $=193/512$ (ii) prob $=[1-2p(1-p)]^k 2p(1-p)$

6. $\phi = \sum_{n=1}^{5} B_n \sin nx \sin\sqrt{25-n^2}\,y$ $\phi=0$

7. (i) $\underline{u} = ((3x^2+y)/y, (x+z)(y,1))$

8 (a)(i) $x+x^2/2+5x^3/6$ (ii) $1-x+x^2$ (iii) $-x^2/2-x^4/12-7x^6/240$

(b)(i) $c_0 = a_0b_0$ $c_1 = a_0b_1+a_1b_0$ $c_2 = a_0b_2+a_1b_1+a_2b_0$ (ii) $c_i = \sum_{j=0}^{i} a_j b_{i-j}$

9. (b)(i) $pk + p(p-1)k^2/100 + \ldots$ (ii) $kp+mq+\frac{1}{200}(k^2p(p-1)+2kmpq+m^2q(q-1))$

10. (b)(ii) $k_2 = 0.482$ $k_4 = 0.464$ (iii) 1.20

11. (i) $\alpha=3$ $\beta=2$ and $\alpha=0$ $\beta=1$ (ii) $y=-1/x^2+1/x$ $y=\sin(\ln x)$

12. (a) $a=(4V/3)^{1/3}=\frac{2b}{3}=2c$ cost $=(36V)^{2/3}/2$

(b) $(-7/2,-7/2)$ max; $(3/2,3/2)$ min; $(-1/2,5/2)$, $(5/2,-1/2)$ saddles