ECONOMICS TRIPOS PART I

Friday 15 June 2007 9 to 12

Paper 3

QUANTITATIVE METHODS IN ECONOMICS

This exam comprises ***four*** *sections. Sections A and B are on* ***Mathematics****; Sections C and D are on* ***Statistics****. You should do the appropriate number of questions from each section. The number of questions to be attempted is given at the beginning of each section.*

Answers from the ***Mathematics*** *and the* ***Statistics*** *Sections must be tied up in separate bundles, with the letter of the Section written on* ***each*** *cover sheet.*

This written exam carries 80% of the marks for Paper 3. Section A carries 24% of the marks, Section B carries 16% of the marks, Section C carries 24% of the marks and Section D carries 16% of the marks.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 x 20 Page booklet	List of statistical formulae
Metric Graph Paper	New Cambridge Elementary Statistical Tables
Rough Work pads	
Tags	Approved calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A - MATHEMATICS

Answer **four** questions.

1 Let A and P be 2×2 matrices defined by

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and let P' be the transpose of P.

(a) Show that the matrix $P'P$ is equal to the identity matrix I, and that the matrix PAP' has all off-diagonal elements equal to zero.

(b) Hence, or otherwise, show that the function

$$f(x, y) = x^2 - xy + y^2$$

attains a minimum at the point $(0, 0)$.

2 (a) Find the solution to the difference equation $x_n = bx_{n-1} + a$ with initial condition $x_0 = 1$.

(b) Comment on the stability properties of the solution to part (a) when $b = -1$ and when $| b |< 1$.

3 Suppose that a monopolist faces the following demand for its output:

$$q = a - bp$$

where q is quantity demanded, p is price and a and b are positive constants, and $a > b$. The average cost of producing q units is given by:

$$AC(q) = 10 + \frac{q}{4}$$

(a) Derive an expression for the monopolist's revenue-maximising level of output.

(b) Derive an expression for the price elasticity of demand in this market and show that demand is unitary elastic at the level of output calculated in (a).

(c) If $a = 20$ and $b = 1$, calculate the monopolist's profit-maximising level of output.

4 Solve the following integrals

(a)
$$\int x^2 \ln x dx$$

(b)
$$\int x\sqrt{(3x+1)}dx$$

(c) A consumer plans to purchase a second-hand car she has seen advertised. She is ignorant of the precise quality of the car, but knows that it can be represented by a continuous random variable α which is uniformly distributed over the range 4 to 10, with 10 denoting the highest possible quality. Write down an integral expression for the expected value of α and evaluate it.

5 Suppose that a firm produces output, Y, according to the production function:

$$Y = A\left(\alpha L^{-\gamma} + \beta K^{-\gamma}\right)^{-\frac{1}{\gamma}}$$

where A, α, β and γ are positive constants.

(a) Determine the returns to scale exhibited by this technology.

(b) Derive expressions for the marginal products of labour and capital.

(c) Hence, or otherwise, derive an expression for the firm's marginal rate of technical substitution.

6 (a) Find the partial derivatives of:

$$z = f(x,y) = (x^{\frac{1}{3}} + y^{\frac{2}{3}})^{\frac{1}{2}}$$

(b) Find the derivative $\frac{dy}{dx}$ of:

$$y = x^{24} \ln\left(\frac{e^{2x}}{x^2 + e^x}\right)$$

TURN OVER

SECTION B - MATHEMATICS

Answer **one** question.

1 Consider the following simple macroeconomic model of a closed economy

$$\begin{aligned}
&Y = C + I + G \\
&C = \bar{C} + c(1-t)Y \\
&I = \bar{I} + aY - br \\
&G = \bar{G} + tY \\
&M_d = \alpha Y - \beta r \\
&M_d = M_s
\end{aligned}$$

where Y, C, I and G respectively denote national income, consumption, investment and government expenditure. t denotes a proportionate tax rate on income, used to finance government expenditure. M_d, M_s and r denote respectively money demand, money supply and the interest rate. $\bar{C}, \bar{I}, \bar{G}$ and the parameters c, a, b, α and β are positive constants, and $a + c < 1$.

(a) Derive a system of two equations, one giving the relationship between r and Y consistent with goods market equlibrium (the IS curve), and the other giving the relationship between r and Y consistent with money market equlibrium (the LM curve).

(b) Derive the determinant associated with this system of equations, and show that it is negative.

(c) Hence, or otherwise, solve for the values of r and Y consistent with macroeconomic equilibrium, and evaluate the effect on the interest rate of an increase in autonomous investment $\bar{I}$.

(d) Evaluate the effect on national income of a decrease in autonomous government expenditure $\bar{G}$. How does the size of the effect depend on the proportionate tax rate t? Explain the intuition behind your answer.

2 A consumer maximises the utility function:

$$U(x, y) = \frac{1}{2} \ln x + \frac{1}{3} \ln y$$

subject to the following budget constraint:

$$p_x x + p_y y = B$$

(a) Write down the Lagrangean expression for this problem and solve for the utility-maximising values x^* and y^* as functions of B, p_x and p_y.

(b) Suppose the consumer's budget B is equal to 50, p_x (the price of x) is 6, and p_y (the price of y) is 5. Solve for x^*, y^* and the value of the Lagrange multiplier.

(c) What is the consumer's utility at the optimal point (x^*, y^*)? How would utility increase if the consumer's budget increased from 50 to 51?

(d) How are your answers to parts (b) and (c) related? Show that your result holds for an arbitrary utility function $U(x, y)$.

TURN OVER

SECTION C - STATISTICS

Answer **four** questions.

1 (a) What is meant by the expected value (or mathematical expectation) of a random variable?

(b) In a lottery, there are 200 prizes of £5, 20 prizes of £25 and 5 prizes of £100. Assuming that 1,000 tickets are to be issued and sold, what ticket price generates a zero profit? (There are no costs of running the lottery.)

(c) If $E(X) = 8$ and $Var(X) = 4$, calculate the expected value and variance of Y if $Y = 3X + 2$.

(d) What is the expected value and the variance of $Z = (X - \mu)/\sigma$, where $\mu = E(X)$ and $\sigma^2 = Var(X)$ [hint: $Z = aX + b$, where $a = 1/\sigma$ and $b = -(\mu/\sigma)$].

2

Year	Business start-ups (in 100,000s)	Business failures (in 10,000s)
1995	6.3	5.2
1996	6.6	5.7
1997	7.0	6.1
1998	6.8	6.1
1999	6.9	5.7
2000	6.7	5.0
2001	6.5	6.0
2002	6.3	8.8
2003	6.6	9.7
2004	7.0	8.6
2005	7.4	7.1
2006	7.7	7.2

Using the data given above, calculate a 95% confidence interval for the population correlation coefficient between business start-ups and business failures.

3 (a) A and B are playing a game which involves each of them tossing 50 coins. A will win the game if she tosses at least 5 more heads than B, otherwise B wins. What is the probability that A will win the game if each tosses a fair coin?

(b) In a certain subject 25% of the students get firsts in their third year. In a college with 10 students in each year, what is the probability that between 20% and 30% get firsts in their third year?

4 The following times are recorded for the annual race around a college in a particular year:

Time recorded (in minutes)	Number of runners recording this time
7-9	2
10-15	4
16-20	8
21-25	4
26-30	1
More than 30	1

(a) Calculate a measure of spread that could be used to compare these results with those of previous years.

(b) Explain the reasons for your choice of measure.

5 The following marks were obtained in a statistics exam:

Men
80, 78, 67, 62, 63, 54, 44, 40, 32, 20;

Women
67, 60, 62, 64, 60, 58.

(a) Test the hypothesis that men and women perform equally well in this exam.

(b) Comment upon your results.

TURN OVER

6 A researcher estimated the following regression equation for Real GDP (Y) and Fixed Capital (K) for Mexico between 1955 and 1974:

$$
\begin{array}{cccccc}
\widehat{\ln Y_t} = & -0.61 & + & 1.01 & \ln K_t & \\
s.e. & (0.23) & & (0.02) & & \\
t\text{-}stat & -2.65 & & 55.12 & & \\
R^2 = .99 & & & & &
\end{array}
\qquad (1.1)
$$

(a) What percentage of the variation in $\ln \hat{Y}_t$ is explained by the equation? Does this seem realistic?

(b) Test the hypothesis of constant returns to scale in the equation (1.1).

(c) The researcher then decides that Employment (L) should be included in the relationship and that constant returns to scale can be assumed in the new model. This leads the researcher to estimate the following equation:

$$
\begin{array}{ccccc}
\ln \widehat{(Y_t/L_t)} = & -0.49 & + & 1.01 & \ln (K_t/L_t) \\
s.e. & (0.12) & & (0.03) & \\
t\text{-}stat & -4.06 & & 28.10 & \\
R^2 = .97 & & & &
\end{array}
$$

Interpret these results.

SECTION D - STATISTICS

Answer **one** question.

1 (a) Briefly discuss the logic of hypothesis testing in the context of statistics.

(b) What assumptions must hold true for the method of ordinary least squares regression to be appropriately applied?

(c) A car maker is examining how petrol consumption is affected by the speed of a given model of car. The data obtained are shown in the table below.

S : Speed (Miles per hour)	E : Economy (Miles per gallon of petrol)
20	55
25	50
30	45
50	39
60	38
70	37

(i) Discuss the likely difficulties in collecting data such as these.

(ii) Carry out a linear regression of either S on E, or of E on S, whichever you think is appropriate to the question.

(iii) Are your results statistically significant?

(iv) The manufacturer is keen to indicate the number of miles per gallon that would be consumed for a constant speed of 40 miles per hour. What is the best estimate of this, based on your results?

(v) What is the number of miles per gallon implied by a speed of zero?

(vi) Calculate the residuals, assuming your results are correct.

(vii) In the light of your result to part (vi), would you revise any of your previous answers?

(viii) Comment on your analysis.

TURN OVER

2 A researcher wishes to seasonally adjust the following series, Y_t :

Table 1

	$Quarter1$	$Quarter2$	$Quarter3$	$Quarter4$
$Year1$	2711	2871	3124	3690
$Year2$	2877	3093	3416	4006
$Year3$	3190	3353	3611	4396
$Year4$	3440	3640	3979	4612
$Year5$	3725	3874	4267	5171

In order to derive the seasonal factors the following model is assumed:

$$Y_t = Z_t(1 + s_t)e^{R_t}, \tag{1.2}$$

$t = 1\ldots..20$ (i.e., t denotes the successive quarters not years), where

$$Z_t = 3000(1 + g)^{t-1}, \tag{1.3}$$

and the seasonal factors s_t only take on four different values, S_1 for quarter 1, S_2 for quarter 2, S_3 for quarter 3 and S_4 for quarter 4 of any year (so that s_t is S_1 for $t = 1, 5, 9, 13, 17$; S_2 for $t = 2, 6, 10, 14, 18$, etc.), and the average of the four seasonal factors is 0. Finally, R_t is random with mean 0.

Estimating g to be 0.02, the researcher determines a series for Z_t. The researcher next uses this series and the original data on Y_t to determine the following estimates of $\ln(Y_t/Z_t)$ to 3 decimal places:

Table 2

	$Quarter1$	$Quarter2$	$Quarter3$	$Quarter4$
$Year1$	-0.096	-0.059	0.006	0.153
$Year2$	-0.116	-0.064	0.016	0.156
$Year3$	-0.092	-0.062	-0.007	0.169
$Year4$	-0.096	-0.059	0.010	0.138
$Year5$	-0.095	-0.076	0.001	0.173

(a) Discuss the specifications given in (1.2) and (1.3).

(b) Reformulate (1.2) in terms of natural logarithms (remembering that $\ln(1 + x) = x$ [approximately] if x is between -1 and 1).

(c) Hence, or otherwise, use the information in Table 1 to determine estimates of the seasonal factors S_1, S_2, S_3 and S_4 to 3 decimal places.

(d) The researcher next carries out a regression analysis of the following model:

$$\ln Y_t = \alpha + \beta(t-1) + \gamma_1 Q_1 + \gamma_2 Q_2 + \gamma_3 Q_3 + \varepsilon_t, \tag{1.4}$$

where ε_t is an error term and

$$\begin{aligned} Q_1 &= (1,0,0,0,1,0,0,0,1\ldots), \\ Q_2 &= (0,1,0,0,0,1,0,0,0,1,\ldots), \\ Q_3 &= (0,0,1,0,0,0,1,0,0,0,1,\ldots). \end{aligned}$$

What estimates of α, β, γ_1, γ_2 and γ_3 would you expect given the results you obtained in (b)?

(e) The researcher estimates equation (1.4) using ordinary least squares regression and obtains the following result:

$$\begin{array}{lllllllll} \widehat{\ln Y_t} = & 8.159 & +0.0198 & (t-1) & -0.257 & Q_1 & -0.222 & Q_2 & -0.153 \; Q_3 \\ s.e. & (0.007) & (0.0004) & & (0.007) & & (0.007) & & (0.007) \end{array}$$

How do these results compare with your answers to (d)?

(f) Comment on the whole exercise, and interpret your findings. How would you do things differently if the data were cyclical?

END OF PAPER