UNIVERSITY OF CAMBRIDGE

ECT1
ECONOMICS TRIPOS PART I

Thursday 13 June 2019 9:00am – 12:00pm

Paper 3

QUANTITATIVE METHODS IN ECONOMICS

Answer **ALL FOUR** questions from Section A, **ONE** question from Section B, **ALL FOUR** questions from Section C and **ONE** question from Section D.

Answers from Sections A and B (Mathematics) and from Section C and D (Statistics) must be written in separate booklets with the letter of the Section written on each cover sheet.

Section A and C each carry 30% of the marks. Section B and D each carry 20% of the marks.

Each question within each section will carry equal weight.

Write your **candidate number** (not your name) on the cover of each booklet.

Candidates are asked to note that there may be a reduction in marks for scripts with illegible handwriting.

If you identify an error in this paper, please alert the **Invigilator**, who will notify the **Examiner**. A **general** announcement will be made if the error is validated.

STATIONERY REQUIREMENTS
20 Page booklet x 2
Rough work pads
Tags

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAMINATION
Calculator – students are permitted to bring an approved calculator.
Durbin Watson and Dickey Fuller Tables

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A – Answer **ALL FOUR** questions from this Section.

1. Consider the matrix

$$L = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

   (a) What geometric linear transformation does L represent?

   (b) Using your answer above, or otherwise, solve the following system of equations

$$\begin{bmatrix} 15 \\ 6 \end{bmatrix} = L^{1215} \begin{bmatrix} m \\ c \end{bmatrix}$$

2. Explain why the partial derivative of the expenditure function with respect to p_i is the Hicksian demand for good i.

3. Suppose a triangle has sides of length $m, 2m,$ and $15.$ What is the maximum possible area of this triangle? Hint: depict the triangle on the Cartesian plane with vertices $(0,0)$, $(15,0)$ and (a,b).

4. Determine which of the following two expressions is the greater:

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \qquad \text{or} \qquad \int_1^{\infty} \frac{dx}{x^2}\ ,$$

and explain your reasoning.

SECTION B – Answer **ONE** question from this Section.

5. $A, B, S,$ and T are positive constants such that $B > A > 0$, $S > 0$, and $T > 0$.

   (a) Determine the set of all $(x, y) \in \mathbb{R}^2_{\geq 0}$ for which the following expression is well-defined
   $$u(x, y) \quad = \quad A \ln(x - S) - B \ln(T - y)$$
   and verify that u represents rational, strictly increasing preferences on that domain.

   (b) Show that these preferences are strictly convex.

   Assume $Sp_x < m < Tp_y$, where p_x and p_y are the unit prices of goods x and y, respectively, and m is the income of a consumer whose preferences over bundles (x, y) are represented by u.

   (c) Show that the consumer's optimal choice problem has a unique solution.

   (d) Compute the consumer's demand function, and show that one of the two goods must be a Giffen good.

6. Suppose you borrow $100,000$ from a bank to finance the purchase of a house. The annual interest rate on this loan is fixed at 12.7%.

   (a) i. What is the monthly interest rate which would lead to the above annual rate if compounded monthly? (Give your answer correct to at least two decimal places.)

   ii. After five years, you would like to have paid off exactly 20% of the principal loan. Calculate your monthly payment m assuming that it remains constant.

   iii. Now, suppose you anticipate the inflation rate to be constant over the next five years, amounting to 0.5% month-on-month. You are willing to increase your monthly payments in line with this rate of inflation. Denoting by m_0 your initial payment at the end of the first month, write the difference equation that relates m_0, debt D_t at the end of period t, and debt D_{t-1} at the end of period $t - 1$.

   (b) Solve the equation you derived in (a)iii to obtain a formula for D_t in terms of D_0, m_0, the interest rate and the inflation rate. Compute m_0 for the above problem.

SECTION C – Answer **ALL FOUR** questions from this Section.

7. X_1 and X_2 are discrete random variables whose joint probability mass function is given in the Table below

		X_2	
		0	1
X_1	0	$^1/_{12}$	$^3/_{12}$
	1	$^2/_{12}$	$^6/_{12}$

(a) Calculate the marginal probability mass functions for X_1 and X_2. Are X_1 and X_2 independent?

(b) Show that there are an infinite number of other joint probability mass functions that have the same marginal distributions.

(c) What restrictions (if any) do the marginal distributions impose on the possible joint probability mass functions?

8. Let A, B, C be events in a sample space Ω. Define A^c to be those elements of Ω not in A. Show using a Venn diagram that

$$P\left(A^c \cap (B \cup C)\right) = P\left(B\right) + P\left(C\right) - P\left(B \cap C\right) - P\left(A \cap C\right) - P\left(A \cap B\right) + P\left(A \cap B \cap C\right)$$

Hence, show the number of integers between 1 and 100 (inclusive) that are divisible by 5 or 7 but not by 3 is 22.

9. A random sample of size $n = 100$ is available from a population with mean μ and known variance $\sigma^2 = 10000$. The following hypothesis test

$$H_0 : \mu \geq 30$$
$$H_1 : \mu < 30$$

is carried out at a 5% significance level. Calculate the power of this test at $\mu = 26$.

10. (a) An economist has taken a random sample of observations $\{x_i, y_i\}$ for $i = 1, 2, \ldots, N$ from the following population regression model

$$y_i = \alpha + \beta x_i + \varepsilon_i \tag{1}$$

She obtains the following estimates of the model parameters by ordinary least squares: $\widehat{\alpha} = 1$ and $\widehat{\beta} = 2$.

Given $\frac{1}{N}\sum(x_i - \bar{x})^2 = 10$ and $\frac{1}{N}\sum(y_i - \bar{y})^2 = 50$, find $\frac{1}{N}\sum(e_i)^2$, where $e_i = y_i - \widehat{y_i}$.

(b) Show that $\sum_{i=1}^{N} e_i = 0$.

(c) Given (1), suppose that $E[\varepsilon_i] \neq 0$. Show that the model can always be written with the same slope, but with a new intercept and error, where the new error has expected value zero.

(d) Which of the following can cause a least squares estimator to be biased? Explain your answer.

- Heteroscedasticity
- Omitting an important variable
- A linear transformation of the X variable

SECTION D – Answer **ONE** question from this Section.

11. A discrete random variable X takes values x_i with probabilities $P(X = x_i)$.

(a) Define $E(X)$ and $E(g(X))$ where $g(\cdot)$ is any function.

(b) Show that if $g(X) = e^{tX}$ where $t \in \mathbb{R}$ then

$$E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} EX^r$$

(c) Now define

$$M(t) = E\left(e^{tX}\right)$$

Assuming that $M(t)$ is differentiable, show that the k^{th} derivative of $M(t)$ (with respect to t) evaluated at $t = 0$ is EX^r, the r^{th} non-central moment of the random variable X.

(d) Calculate $M(t)$ in the case where X is a Bernoulli random variable which takes the value 1 with probability p and the value 0 with probability $(1 - p)$. Verify that the first two derivatives of the function you obtain imply for a Bernoulli random variable X that $E(X) = E(X^2) = p$.

(e) Prove that if $X_1, X_2, \ldots, X_n$ are n independent random variables and $Y = X_1 + X_2 + \ldots + X_n$ then

$$E\left(e^{tY}\right) = \prod_{j=1}^{n} E\left(e^{tX_j}\right)$$

(f) Hence, or otherwise, show that if $X_1, X_2, \ldots, X_n$ are independent, identically distributed Bernoulli random variables, then the random variable $Y = X_1 + X_2 + \ldots + X_n$ has its first two moments given by:

$$E(Y) = np$$
$$E\left(Y^2\right) = n(n-1)p^2 + np$$

and thus for a binomial random variable $Y = B(n, p)$ (giving the number of successes in n independent Bernoulli trials each with probability of success p) we have

$$E(Y) = np$$
$$Var(Y) = np(1-p)$$

You may use without proof the fact that the Taylor series expansion for e^z is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

for all z. You may also assume that, if necessary, expectation and summation operators can be interchanged.

12. An analyst has a sample of data on years of education (X_i) and hourly wage (Y_i) for $i = 1, \ldots, N$ observations.

(a) A scatterplot of the data reveals

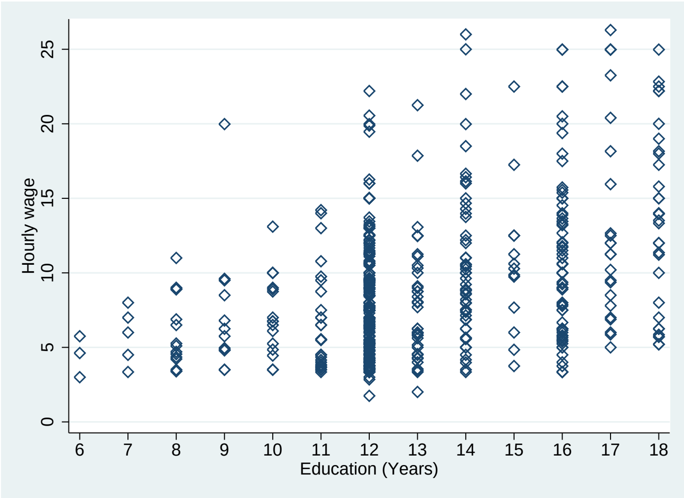

Figure 1: A Scatterplot of Wages versus Education

If an analyst were to fit an ordinary least squares regression line through this data, which of the Gauss Markov assumptions are likely to be violated? Explain your answer.

(b) Suppose you were asked to construct a regression model for Y without regard to X. We may write such as model as

$$Y_i = \alpha + \varepsilon_i \quad i = 1, \ldots, N$$

Show that the ordinary least squares (OLS) estimator is given by $\widehat{\alpha} = \frac{1}{N} \sum_{i=1}^{N} Y_i$, the sample mean of hourly wages.

(c) Another analyst utilises the same data but only has available $N - 1$ observations. In this instance, in what sense is the OLS estimator inferior to that used in (b)?

(d) An analyst utilises the following linear regression model including X_i.

$$Y_i = \alpha + \beta X_i + \varepsilon_i \quad i = 1, \ldots, N$$

Given $\widehat{\beta} = \frac{\sum_{i=1}^{N} X_i^* Y_i}{\sum_{i=1}^{N} (X_i^*)^2}$, where $X_i^* = X_i - \bar{X}$, $\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$, $\sum_{i=1}^{N} X_i^* = 0$ and

$$\sum_{i=1}^{N} X_i^* X_i = \sum_{i=1}^{N} (X_i^*)^2$$

show that $E(\widehat{\beta}) = \beta$.

END OF PAPER