ECONOMICS TRIPOS PART I

MOCK EXAMINATION 2020

Paper 3 QUANTITATIVE METHODS IN ECONOMICS - 3 HOUR EXAMINATION

Answer **ALL FOUR** questions from Section A**, ONE** question from Section B, **ALL FOUR** questions from Section C, and **ONE** question from Section D.

Section A and C each carry 30% of the total marks for this paper. Sections B and D each carry 20% of the total marks.

Each question within each section will carry equal weight.

Durbin Watson and Dickey Fuller Tables

Students are permitted to use an approved calculator.

SECTION A – Answer **ALL FOUR** questions from this Section.

1. The function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) \quad = \quad \begin{cases} \frac{\exp(-2/x)}{x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine whether f is continuous or not.

(b) Determine whether f is bounded or not.

(c) If they exist, determine all points where f attains its maximum or minimum values.

Make sure you justify your answer in each part.

2. (a) Define what it means for a function from $\mathbb{R}^n$ to $\mathbb{R}$ to be convex.

(b) True or false: if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are both convex, then $f + g$ is also convex. If true, prove it. If not, explain why not.

(c) True or false: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are both convex, then $f \circ g$ is also convex. If true, prove it. If not, explain why not.

3. Consider the problem

$$\max_{x,y,z} \quad xy \exp(z/2020) \qquad \text{s.t.} \quad x^2 + y^2 + z \leq 2019 \;\; \text{and} \;\; z \geq 0$$

(a) Show that the problem must have a solution.

(b) Compute all solutions of the above problem.

4. Using Taylor polynomials, find a rational number which is at most 10^{-11} away from $8.002^{1/3}$.

(*Hint: You can begin with showing that a second order Taylor polynomial should be sufficient.*)

SECTION B – Answer **ONE** question from this Section.

5. Consider the objective function $u : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$u(x, y) \quad = \quad -(x-2)^2 - (y-3)^2$$

(a) Does the problem

$$\max_{x,y} \; u(x, y) \qquad \text{subject to} \quad 52 \leq (x-4)^2 + y^2 \leq 208$$

have a solution? Explain why or why not.

(b) Does the problem

$$\min_{x,y} \; u(x, y) \qquad \text{subject to} \quad 52 \leq (x-4)^2 + y^2 \leq 208$$

have a solution? Explain why or why not.

(c) Assuming the above problems do have solutions, determine whether there exists any interior solution. If they exist, compute all such solutions.

(d) Using the Lagrangian approach, compute all solutions to the above optimisation problems.

(e) Give an estimate for the minimum value that u takes subject to the constraint

$$51.999 \leq (x-4)^2 + y^2 \leq 208.002$$

and explain briefly if your estimate is higher or lower than the actual minimum value attained.

6. Suppose that in a market for a particular good, a market maker sets a price in each period. Based on this price, aggregate demand and supply materialise (but markets do not necessarily clear). If in any period the demand exceeds the supply, say by Δ, then the market maker increases the next period price by $\gamma\Delta$, where $\gamma > 0$ is a constant. If, on the other hand, the supply exceeds the demand by Δ, then the market maker decreases the next period price by $\gamma\Delta$.

(a) Denoting demand and supply functions by D and S, respectively, and the price in period t by p_t, write down the difference equation that captures the price dynamics in this model.

(b) Suppose the price elasticity of demand at price p is given by $ap/(A + ap)$, and the price elasticity of supply at price p is given by $bp/(B+bp)$. Moreover, when $p = 1$, demand and supply are equal to $D(1) = 2A + 2a$ and $S(1) = 2B + 2b$, respectively. Derive the functions D and S.

(c) Solve the difference equation to obtain a formula for p_t in terms of the initial value p_0 and the parameters A, a, B, b, γ.

(d) Obtain the conditions under which the price converges.

(e) Comment on the nature of convergence and what the limit price means.

SECTION C – Answer **ALL FOUR** questions from this Section.

7. Let

$$P(X=n, Y=k) = {}^{n}C_{k}\left(\frac{1}{6}\right)^{k}\left(\frac{1}{3}\right)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ be the joint probability mass function of the discrete random variables X and Y where ${}^{n}C_{k} = \frac{n!}{k!(n-k)!}$

What is the conditional distribution of Y given that $X = n$? Verify that your answer is a probability mass function.

You may use without proof that $(a+b)^n = \sum_{k=0}^{n} {}^{n}C_{k} a^k b^{n-k}$.

8. A geometric random variable X measures the number of trials until success when each trial is an independent Bernoulli with probability of success p. The probability mass function is

$$P(X=k) = p(1-p)^{k-1} \qquad k = 1, 2, \ldots$$

(a) Show that $E(X) = \frac{1}{p}$

(b) Alice and Bob play a game. First, Alice rolls a fair 6-sided dice. Then Bob rolls a similar dice, if necessary multiple times, until he rolls a number equal or higher than Alice's. What is the expected number of rolls that Bob will take?

You may use without proof that $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$ for $0 < x < 1$.

9. A random variable X is uniformly distributed on the interval $[0, \theta]$ where θ is an unknown positive parameter. The probability density function is

$$f(x) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & \text{elsewhere} \end{cases}$$

A single realisation of X is drawn and the statistician decides to use it as an estimator of θ. That is, the statistician chooses the estimator $\hat{\theta} = X$.

(a) Calculate the bias and mean squared error (MSE) of $\hat{\theta}$.

(b) As an alternative consider the estimator $\tilde{\theta} = k\hat{\theta}$ where k is some number. Is it possible to find a value of k such that

$$MSE\left(\tilde{\theta}\right) < MSE\left(\hat{\theta}\right)$$

Justify your answer.

10. (a) Carefully explain what you understand by a two-tailed and one-tailed hypothesis test.

(b) Under what conditions is a one-tailed test appropriate?

(c) You are given the following data based on a random sample from two populations, $\mathcal{A}$ and $\mathcal{B}$.

$\bar{x}_{\mathcal{A}} = 50.121$, $\widehat{\sigma}_{\bar{x}_{\mathcal{A}}} = 1.080$, and $N_{\mathcal{A}} = 91$

$\bar{x}_{\mathcal{B}} = 55.991$, $\widehat{\sigma}_{\bar{x}_{\mathcal{B}}} = 0.779$, and $N_{\mathcal{B}} = 109$

For $D = \mu_{\mathcal{A}} - \mu_{\mathcal{B}}$, and assuming $\sigma^2_{\mathcal{A}} = \sigma^2_{\mathcal{B}}$, test the hypothesis that $D = 0$ against a two-sided alternative.

(d) Consider the following null and alternative hypotheses about D.

- $H_0 : D = 0 \quad H_1 : D > 0$
- $H_0 : D = 0 \quad H_1 : D < 0$
- $H_0 : D = 0 \quad H_1 : D \neq 0$

i. Explain why the test statistic, say $\tilde{t}$, is the same for all tests?

ii. Assuming $\tilde{t} \sim t_{\text{df}}$, compute the p-value for all tests, where df is the degrees of freedom you used in (c).

iii. Explain the relationship between the p-values you calculated in ii.

(e) Explain why the power of the test is

i. Increasing in the sample size n.

ii. Increasing in the value under the alternative μ_1.

iii. Decreasing in σ.

SECTION D – Answer **ONE** question from this Section.

11. Liquid waste produced by a factory is removed once a week. The weekly volume of waste in thousands of gallons is a continuous random variable with probability density function

$$f(x) = \begin{cases} Ce^{-x} & \text{if} \quad 0 < x \leq \ln\left(\frac{8}{3}\right) \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of C?

(b) During the week the waste is stored in a tank. How large in thousands of gallons (to 2 decimal places) must the tank be to ensure the probability of overflow during a week is below 5%?

The storage tank manufacturer produces tanks in size 0.8 and 0.9 thousand gallons. The factory owner orders a 0.9 thousand gallon tank. The cost (in £000s) of removing $x > 0$ thousand gallons of waste at the end of the week is $0.5x$. Additional costs $8.55 + 10z$ are incurred when the capacity of the storage tank is not sufficient and an overflow of $z > 0$ thousand gallons of waste occurs during the week.

(c) What is the expected value of the weekly costs?

The factory owner suspects the storage tank manufacturer actually installed a smaller tank. Over a 104 week period she records 9 overflows.

(d) Test at the 5% level the hypothesis that the tank is of the correct capacity. (You may assume the sample size is sufficiently large that the normal approximation holds.)

(e) What would the power of the test you carried out in (d) be against the hypothesis that the tank is actually of the smaller capacity?

12. Suppose we are interested in measuring the impact of textbooks on learning.

Let Y_i^T denote the average test score of children in a school i if the school has textbooks and Y_i^C the test score if school i has no textbooks.

TB_i is a dummy variable equal 1 if school i receives textbooks.

(a) Out of 100 schools, 50 are randomly chosen to receive the textbooks.

The average effect of the textbooks on learning can be estimated by

$$\widehat{D} = \hat{E}[Y_i|TB_i = 1] - \hat{E}[Y_i|TB_i = 0] \tag{1}$$

where Y_i denotes the observed outcome, $\widehat{D}$ the differences in means, and $\hat{E}$ the sample average.

Another analyst seeks to estimate the difference in means using the following equation

$$Y_i = \alpha + \beta TB_i + \varepsilon_i, \tag{2}$$

where ε_i denotes an error term.

By interpreting the ordinary least squares (OLS) estimators $\widehat{\alpha}$ and $\widehat{\beta}$, explain how she might do this.

(b) Outline the assumptions of the OLS regression model. Are they likely to be met in this instance?

(c) If textbooks are randomly allocated to schools, carefully explain why $\widehat{D} = \widehat{\beta}$.

(d) You are now told that schools were selected to receive textbooks based upon existing performance.

How might this impact your estimate of β using equation (2)? How might you modify equation (2)?

END OF PAPER