ECT1
ECONOMICS TRIPOS PART I

Thursday 1 June 2023 9.00am – 12.00pm

Paper 3

QUANTITATIVE METHODS IN ECONOMICS

Answer **ALL FOUR** questions from Section A, **ONE** question from Section B, **ALL FOUR** questions from Section C, and **ONE** question from Section D.

Answers from Sections A and B (Mathematics) must be written in one booklet; answers from Sections C and D (Statistics) must be written in a separate booklet. Write the letters of the sections on each cover sheet.

Sections A and C each carry 30% of the total marks for this paper. Sections B and D each carry 20% of the total marks.

Each question within each section will carry equal weight.

Write your **Blind Grade Number** (not your name) on the cover of each booklet.

Candidates are asked to note that there may be a reduction in marks for scripts with illegible handwriting.

If you identify an error in this paper, please alert the **Invigilator**, who will notify the **Examiner**. A **general** announcement will be made if the error is validated.

STATIONERY REQUIREMENTS
20 Page booklet x 2
Rough work pads
Tags

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAMINATION
Calculator - students are permitted to bring an approved calculator
New Cambridge Elementary Statistical Tables

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A – Answer **ALL FOUR** questions from this Section.

A1 A real-valued function $f(x)$ is defined for $x \in \mathbb{R}$ as follows

$$f(x) = \begin{cases} \exp(x) & \text{for } x \leq 0 \\ x + 1 & \text{for } x > 0 \end{cases}$$

(a) Show that $f(x)$ is differentiable everywhere, and that the derivative $f'(x)$ is a continuous function.

(b) Identify the range of function f (that is, the set of values attained by f). Denoting this range by D, explain why f has an inverse function g defined on D.

(c) Derive an explicit expression for g and identify its range.

A2 Let $S \subseteq \mathbb{R}$ be the set of all solutions to the equation $\sin x = 0$.

(a) Show that S is a closed set.

(b) Show that S is not a compact set.

A3 Let A be the 2×2 matrix obtained by multiplying the column vector $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with its transpose $a' = \begin{pmatrix} 1 & 1 \end{pmatrix}$. That is

$$A = aa'.$$

(a) Evaluate the matrix A^{10}.

(b) What geometric linear transformation does A represent?
(Hint: recall that for any 2×1 vector b, we have $a'b = \|a\| \, \|b\| \cos\beta$, where β is the angle between a and b).

A4 A profit-maximising firm's cost function is

$$c(q) = q + \alpha(1 - q)^2$$

where $q \in \mathbb{R}_{\geq 0}$ is the amount of units produced and α is a constant greater than $\frac{1}{2}$. The unit price of its output is p, also a constant.

(a) Show that the firm's optimal choice of q is positive.

(b) How does the firm's optimal profit change in response to small changes in α?

SECTION B – Answer **ONE** question from this Section.

B5 Consider a household that consists of two people, A and B. Both A and B consume only two goods: good_1 and good_2. The unit prices of both goods are equal to 1. Person A's utility from consuming x_A units of good_1 and y_A units of good_2 equals

$$u_A(x_A, y_A) = 3 - (x_A - 10)^2 - (y_A - 5)^2.$$

Person B's utility from consuming x_B units of good_1 and y_B units of good_2 equals

$$u_B(x_B, y_B) = \frac{1}{2} x_B y_B.$$

The income of A is 15 and the income of B is 20.

(a) Find the unconstrained maximum of function u_A.

(b) Assuming A and B do not share their incomes and are both utility maximisers, how much of each of the good will be consumed by A and by B?

(c) Suppose now that A and B pool their incomes and try to maximise the sum of their utilities. Argue that at the optimum, all the marginal utilities (that is, $\partial u_A/\partial x_A, \partial u_A/\partial y_A, \partial u_B/\partial x_B$, and $\partial u_B/\partial y_B$) will be non-negative.

(d) In the setting of (c), find the optimal consumption bundle $(x_A^*, y_A^*, x_B^*, y_B^*)$.

B6 A profit-maximising firm uses two divisible inputs, K and L, to produce a divisible output. The firm's resulting cost function is

$$C(r, w, y) = y(r + \sqrt{rw} + w)$$

where r and w are the unit prices of K and L, respectively, and y is the amount of output produced.

(a) We define the firm's technical rate of substitution (TRS) as the rate at which it should increase one input in response to a decrease in the other input if the firm aims to keep production constant. Evaluate the firm's TRS when the firm is optimising. Explain your answer.

(b) Derive a production function $f(K, L)$ which leads to the above cost function.

SECTION C – Answer **ALL FOUR** questions from this Section.

C7 A gambler is offered a sequence of betting opportunities where each bet has probability p of winning (and probability $1-p$ of losing). Successive bets are independent events.

With wealth W, if they choose to bet a fraction f of that wealth and if the bet wins, then their wealth grows to $(1+bf)W$; if the bet loses, then their wealth drops to $(1-af)W$, where a and b are both positive constants.

(a) Show that if they bet the same fraction (f) of their wealth in each round of betting, and if they start with initial wealth 1, then the expected value of log wealth after n gambles is

$$\mathrm{E}\left(\ln\left(W\right)\right) \quad = \quad np\ln\left(1+bf\right)+n\left(1-p\right)\ln\left(1-af\right)$$

(b) Suppose, again, that the gambler will bet a fixed fraction of their wealth in each round of betting. Show that to maximise log wealth after n bets, the fraction f^* they choose to bet in each round is given by

$$f^* \quad = \quad \frac{p}{a}-\frac{1-p}{b}$$

(c) A fair bet is defined as one where the expected return is zero. How much will the gambler (who is trying to maximise expected log wealth) stake on a fair bet? If they were to commit to betting a fixed fraction of their wealth in each round of a sequence of bets, what fraction would they choose? Justify your answer.

C8 My fund manager promises they will deliver an average return on investment of 0.5% per month. Over the last 60 months their actual returns have averaged 0.45% per month with a standard deviation (expressed in the s_{n-1} notation) of 0.1. They say this has just been a run of bad luck. Should I fire them for failing to maintain their claimed performance? Justify your answer, being clear about any assumptions you have made.

C9 (a) A fair coin is tossed until a head appears. If a head appears on the i^{th} toss, a ball is selected at random from 5^i balls of which 3^i are red.

i. Find the probability that a red ball is selected.

ii. Given that a red ball was selected, find the probability that a head first appeared on the third toss.

(b) A biased coin is tossed repeatedly and the outcomes are independent. Each time, the probability of heads is 25%. Evaluate the probability that it takes exactly 36 tosses to accumulate 9 heads.

C10 A researcher interested in understanding how the weight (w) of individuals changes over time, writes down the following regression model

$$w_i^{2011} = \alpha + \beta w_i^{2004} + \varepsilon_i \tag{1}$$

where w^{2011} (respectively, w^{2004}) denotes the weight (in pounds) of individuals in 2011 (respectively, 2004).

(a) Based on a sample of 500 individuals, the estimated regression equation is presented below

$$\widehat{w}_i^{2011} = \underset{(0.000)}{17.42} + \underset{(0.000)}{0.97}\, w_i^{2004} \qquad R^2 = 0.71 \tag{2}$$

Note: p-values are in parentheses

(i) Provide an interpretation of the regression equation (2).

(ii) Test the $H_0 : \beta = 0$ against a two-sided alternative.

(b) The summary statistics in Table 1 indicate that, on average, the respondents put on 13 pounds over the period 2004-2011.

Table 1: Summary Statistics

Variable	Mean	Std. Dev	Min	Max
w^{2004}	169.7	40.3	95	330
w^{2011}	182.7	46.7	95	370

Using the results from (a), comment on whether this change is due to the relatively heavy individuals becoming even heavier, or due to a general increase in weight.

(c) What do we learn from $R^2 = 0.71$?

(d) Show that

$$\sum_i (w_i - \overline{w})^2 = \sum_i (\widehat{w}_i - \overline{w})^2 + \sum_i \widehat{\varepsilon}_i^2 \tag{3}$$

where $\widehat{\varepsilon}_i^2 = (w_i - \widehat{w}_i)^2$, and consequently derive an expression for R^2 in terms of w_i^{2011} and w_i^{2004}.

SECTION D – Answer **ONE** question from this Section.

D11 You have a sample of observations $\{y_i, x_i\}$, $i = 1, \ldots, N$, generated by

$$y_i = x_i \beta + \varepsilon_i$$

where β is an unknown parameter, x_i are a set of fixed numbers, and ε_i are IID random variables from a distribution with $\mathrm{E}(\varepsilon_i) = 0$ and $\mathrm{E}(\varepsilon_i^2) = \sigma^2$.

To estimate β it is proposed to use an estimator

$$\widetilde{\beta} = \sum_{i=1}^{N} w_i y_i$$

where the numbers w_i are weights to be determined (that may depend on x_i).

(a) Calculate the bias and variance of $\widetilde{\beta}$. (You should give your answer in terms of w_i, x_i, β and σ^2.)

(b) Define the mean squared error (MSE) of an estimator.

(c) Show that

$$MSE\left(\widetilde{\beta}\right) = \left[\left(\sum_{i=1}^{N} w_i x_i - 1\right)\beta\right]^2 + \sum_{i=1}^{N} w_i^2 \sigma^2$$

(d) Show that if we impose that $\widetilde{\beta}$ is unbiased and then minimise the MSE of $\widetilde{\beta}$, then we obtain as optimal weights

$$w_i = \frac{x_i}{\sum_{j=1}^{N} x_j^2}$$

(e) Show that to minimise the MSE of $\widetilde{\beta}$ (with no restriction on bias), the weights w_i should now be chosen as

$$w_i = \frac{\beta^2 x_i}{\sigma^2 + \beta^2 \sum_{j=1}^{N} x_j^2}$$

(f) Comment on the practicality of the estimator in (e) compared to that in (d).

D12 A variable Y depends on a nonstochastic variable X through the relationship

$$Y_i = \beta_1 + \beta_2 X_i + u_i \tag{4}$$

where β_1 and β_2 are unknown parameters; $\mathrm{E}(u_i|X_i) = 0$ and $\mathrm{Var}(u_i|X_i) = \sigma^2$ for all i; $\mathrm{Cov}(u_i, u_j) = 0$ for all $i \neq j$.

(a) Given a sample of n observations, an analyst decides to estimate β_2 using the expression

$$\widehat{\beta}_2 = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2} \tag{5}$$

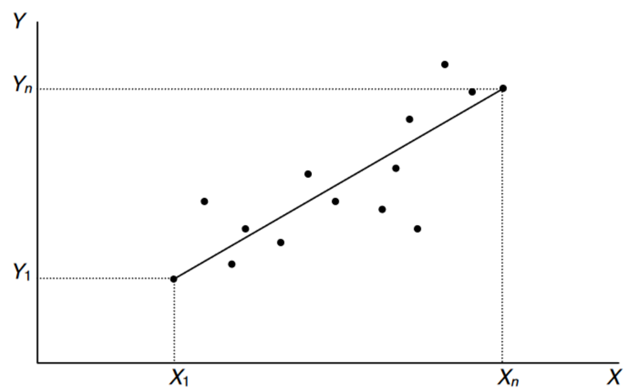
(i) Demonstrate that $\widehat{\beta}_2$ is in general a biased estimator of β_2.

(ii) Comment on whether it is possible to determine the sign of the bias in $\widehat{\beta}_2$.

(iii) Demonstrate that $\widehat{\beta}_2$ is unbiased if $\beta_1 = 0$ or $\bar{X} = 0$.

(b) On seeing the scatter of points on Y and X in the figure below, another analyst attempts to estimate the slope by drawing a line connecting the first and last observations, and dividing the increase in height by the horizontal distance. This estimator is given by

$$\widetilde{\beta}_2 = \frac{Y_n - Y_1}{X_n - X_1} \tag{6}$$



(i) Find an expression for the expected value of $\widetilde{\beta}_2$.

(ii) Which of the Gauss Markov assumptions implies the unbiasedness of $\widetilde{\beta}_2$?

(c) It can be shown that the population variance of $\widetilde{\beta}_2$ in (6) exceeds that of the OLS estimator.

In what sense does this confirm the Gauss Markov theorem?

END OF PAPER