UNIVERSITY OF
CAMBRIDGE

ECT1
ECONOMICS TRIPOS PART I

Friday 31 May 2024   9.00am-12.00pm

Paper 3

QUANTITATIVE METHODS IN ECONOMICS

Answer **ALL FOUR** questions from Section A, **ONE** question from Section B, **ALL FOUR** questions from Section C, and **ONE** question from Section D.

Answers from Section A and B (Mathematics) must be written in one booklet; answers from Section C and D (Statistics) must be written in a separate booklet. Write the letters of the Sections on each cover sheet.

Section A and C will each carry 30% of the total marks for this paper. Section B and D each carry 20% of the total marks.

Each question within each section carries equal weight.

Write your **Blind Grade Number** (not your name) on your answers.

Candidates are asked to note that there may be a reduction in marks for scripts with illegible handwriting.

If you identify an error in this paper, please alert the Invigilator, who will notify the **Examiner**. A general announcement will be made if the error is validated.

STATIONERY REQUIREMENTS
20 Page booklet $\times 2$
Rough work pads
Tags

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAMINATION
Calculator – students are permitted to bring an approved calculator.
New Cambridge Elementary Statistical Tables

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

A1 Let supply of and demand for a good be described by the following functions of price $P \geq 0$:

$$S = \ln(1 + P), \quad D = t - P.$$

In the above demand equation, $t \geq 0$ is time. Note that the demand is growing over time.

(a) Assuming that in the equilibrium $S = D$, establish a mathematical relationship between t and the corresponding equilibrium price P_e by finding t as a function of P_e. Argue that this function is invertible. That is, for each $t \geq 0$, there exists one and only one equilibrium price $P_e(t) \geq 0$.

(b) Express $\frac{\mathrm{d}}{\mathrm{d}t}P_e(t)$ as a function of $P_e(t)$. Evaluate $\frac{\mathrm{d}}{\mathrm{d}t}P_e(t)$ at the moment of time when $P_e(t) = 1$. [Hint: you are welcome to use either the rule for the differentiation of the inverse function or the techniques for evaluating the derivative of an implicit function.]

A2 (a) Let $a > 1$ be a real number. Using integration by parts or otherwise, express the following integral:

$$\int_1^a \ln(x)\mathrm{d}x$$

as an explicit function of a.

(b) Find the derivative of the following function:

$$f(x) = (\ln x)^x, \quad x > 1.$$

[Hint: you might want, first, to express $f(x)$ in the form $e^{g(x)}$.]

A3 Consider a function

$$f(x, y) = x + y + \frac{1}{xy},$$

defined for all real x, y such that $x \neq 0$ and $y \neq 0$.

(a) Find all critical points of this function and determine their types (local minimum, local maximum, or saddle point).

(b) Are there any points of global minimum among the critical points you found in (a)? Provide a clear argument to support your answer.

A4 Consider a matrix

$$A = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

(a) Using the Lagrange multiplier method, find the maximum of the determinant of A subject to the constraint that $x^2 + y^2 = 1$.

(b) Describe the set of points $(x, y) \in \mathbb{R}^2$ such that A is a positive definite matrix. Is this set open? Is it convex? Explain.

SECTION B

B5 A profit maximising firm employs labour (L) to produce yogurt (Y) from milk (M). The firm's production function is

$$Y = f(L, M; T) = Te^{-(T-40)^2/10} L^{1/4} M^{1/2},$$

where $T \geq 0$ is the temperature (in Celsius degrees) maintained during the production process. The exogenously given prices of yogurt, labour and milk are, respectively, $p = 2$, $w = 16$ and $r = 1$.

(a) Suppose that the firm maintains the temperature $T = 40$ during production. Find the optimal values of L and M, and the optimal profit that the firm will make.

(b) Use the envelope theorem to approximate the change in the optimal profit if the temperature T goes down from 40 to 39.9.

(c) What is the optimal temperature T that the firm should try to maintain during the production process? Explain.

(d) Suppose that $L = 4^4 = 256$ and $M = 100$. Find the first order Taylor approximation around $T = 40$ for the function of T given by $f(256, 100; T)$. Assuming that the second order derivative $\frac{\mathrm{d}^2}{\mathrm{d}T^2} f(256, 100; T)$ is negative and decreasing on the segment $T \in [39.9, 40]$, derive an upper bound on the absolute error of your approximation at $T = 39.9$.

B6 Let $x(t)$ and $y(t)$ be, respectively, the populations of rabbits and foxes on a field at time $t \geq 0$. Suppose that $x(t)$ and $y(t)$ are related via the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) - \beta y(t)x(t) + \gamma,$$

where α, β and γ are some real numbers. For simplicity, we will allow $x(t)$ and $y(t)$ to be non-integer numbers.

(a) Explain why it is reasonable to expect that $\alpha > 0$ and $\beta > 0$.

(b) Suppose that $\alpha = 3$, $\beta = 1$, $\gamma = 1$, the initial population of rabbits is $x(0) = 100$, and the population of foxes stays constant at $y(t) = 2$. Find $x(t)$ as a function of time t.

(c) Suppose now that all assumptions of (b) hold, except that at time $t = 1$, $y(t)$ increases from 2 to 4, and stays at 4 forever after that. At what level will the population of rabbits stabilize? At what time will it reach the level $x(t) = 15$? Explain.

(d) Finally, suppose that $\alpha = 3$, $\beta = 1$, and $x(0) = 100$ as above. However, $\gamma = 0$ and $y(t) = 3 - \cos t$. Find $x(t)$ as a function of t. Between what maximum and minimum values will $x(t)$ fluctuate?

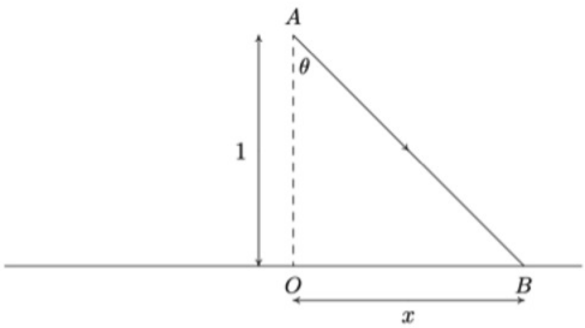
SECTION C

C7 A person uses their car 30% of the time, walks 30% of the time and rides the bus 40% of the time as they go to work. They are late 3% of the time when they drive; they are late 10% of the time when they walk; and they are late 7% of the time they take the bus.

(a) What is the probability they took the bus if they were late?

(b) What is the probability they walked if they were on time?

C8 A footballer takes a penalty from the point A located at a distance 1 from the goal line and kicks the ball at an angle θ as shown. The ball crosses the goal line at point B located at a distance x from the centre of the goal (point O).



If θ is uniformly distributed on $0 < \theta < \frac{\pi}{2}$ find and sketch the pdf of the random variable X measuring the distance OB. [Hint: it may be useful to recall the derivatives of the inverse trigonometric functions:

$$\begin{aligned} \frac{d}{dz}\arcsin(z) &= \frac{1}{\sqrt{1-z^2}} \quad \text{for} \quad -1 < z < 1, \\ \frac{d}{dz}\arccos(z) &= -\frac{1}{\sqrt{1-z^2}} \quad \text{for} \quad -1 < z < 1, \quad \text{and} \\ \frac{d}{dz}\arctan(z) &= \frac{1}{1+z^2} \quad \text{for} \quad -\infty < z < \infty.] \end{aligned}$$

C9 The following table gives data on log wages for a random sample of 3294 US individuals:

	Male	Female
Average	1.6896	1.4748
SD	0.5974	0.6308
No of Obs	1725	1569

(a) Test the hypothesis that average log wages are the same for males and females. Be clear about any assumptions you make.

(b) Is this test informative about whether average wages are the same? Why or why not?

(c) Explain how you would construct a confidence interval for the variance of log wages for females.

C10 You believe that the percentage rate of price inflation in an economy, p, depends on the percentage rate of wage inflation, w, according to the linear equation

$$p = \alpha + \beta w + u,$$

where α and β are unknown parameters and u is an error term, which is normally distributed with a zero mean and is independent from w.

The model, fitted by OLS based on $n = 20$ observations, is given by

$$\hat{p} = \underset{(0.05)}{-1.21} + \underset{(0.10)}{0.82}w,$$

with estimated standard errors in the parentheses.

(a) An analyst seeks to test the following hypothesis:

$$H_0 : \beta = \beta^* \quad \text{against} \quad H_1 : \beta \neq \beta^*,$$

where β^* is some real number.

(i) State the decision rule for H_0 in terms of the random variable $Z = (\hat{\beta} - \beta^*)/\hat{\sigma}_{\hat{\beta}}$, where $\hat{\sigma}_{\hat{\beta}}$ denotes the estimated standard error of the OLS estimator $\hat{\beta}$.

(ii) At the 5% significance level find the set of values of β^* that will not lead to a rejection of H_0.

(b) (i) Explain the distinction between the use of a confidence interval and a hypothesis test.

(ii) Outline the advantages of the use of a confidence interval over a hypothesis test.

(c) The null hypothesis $H_0 : \beta = 1$ states that wage inflation is fully reflected in price inflation.

(i) Why might an analyst wish to test $H_0 : \beta = 1$ against a one-sided alternative $H_1 : \beta < 1$, rather than conducting a two-sided test?

(ii) Using the 5% significance level, undertake a one-sided test $H_1 : \beta < 1$ and a two-sided test $H_1 : \beta \neq 1$, and compare your findings.

SECTION D

D11 (a) A bag contains 3 black balls, 2 white balls and 1 red ball. Two balls are drawn at random from the bag (without replacement). Let X be the number of black balls selected and Y the number of white. The joint probability mass function $p_{XY}(x, y)$ of the discrete bivariate random variable (X, Y) is given by

		X		
		$x = 0$	$x = 1$	$x = 2$
Y	$y = 0$	0	1/5	1/5
	$y = 1$	2/15	2/5	0
	$y = 2$	1/15	0	0

(i) Derive the probability of selecting a pair that contains at most one black ball and at least one white ball.

(ii) Derive the probability of selecting a pair that contains exactly one white ball given that it contains exactly one black ball.

(iii) Derive the probability of selecting a pair that contains exactly one black ball given that it contains exactly one white ball.

(b) Now suppose that a discrete bivariate random variable (X, Y) has the following probability mass function:

		X		
		$x = 0$	$x = 1$	$x = 2$
Y	$y = 0$	1/12	0	1/12
	$y = 1$	1/12	5/12	0
	$y = 2$	1/12	1/12	1/6

(i) Find the value of the covariance $Cov(X, Y)$.

(ii) Prove that X and Y are not independent.

(iii) Find the value of the correlation coefficient ρ_{XY}.

(c) Suppose Y_1 and Y_2 are independent random variables, each distributed uniformly on the segment $[0, 1]$. Use Chebyshev's inequality to find an upper bound on

$$Prob\left(|Y_1 + Y_2 - 1| \geq \frac{1}{2}\right).$$

Note: Chebyshev's inequality states that for a random variable X with $E(X) = \mu$ and $Var(X) = \sigma^2$, we have

$$Prob(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

D12 (a) Assume we have a random sample $\{(x_i, y_i) : i = 1, 2, ..., n\}$ taken from the joint distribution of random variables (x, y) satisfying the following population model:

$$y = \beta_0 + \beta_1 x + u,$$

where β_0 and β_1 are population parameters and u is an error term. Hence,

$$y_i = \beta_0 + \beta_1 x_i + u_i, \quad i = 1, 2, \ldots, n,$$

where u_i is the error for observation i. Show that

$$\sum_{i=1}^{n}(x_i - \bar{x}) = 0 \quad \text{and} \quad \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n}(x_i - \bar{x})y_i,$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

(b) Using your results from (a), show that the OLS estimator of β_1 can be written as

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})y_i}{SST_x},$$

where $SST_x = \sum_{i=1}^{n}(x_i - \bar{x})^2$.

(c) (i) Using your answer to (b), show that $\hat{\beta}_1$ can be written as

$$\hat{\beta}_1 = \beta_1 + (1/SST_x)\sum_{i=1}^{n} d_i u_i,$$

where $d_i = x_i - \bar{x}$.

(ii) Stating sufficient assumptions, show that the conditional variance $Var(\hat{\beta}_1 | x_i, i = 1, \ldots, n)$ equals σ^2/SST_x, where $\sigma^2 = Var(u)$.

(d) Stating sufficient assumptions, show that $\hat{\beta}_1$ is an unbiased estimator of β_1.

END OF PAPER