A Tutorial on Simple First Order Differential Equations
(for Economics Part I Paper 3)

Corrections to Dr Ian Rudy (http://www.robinson.cam.ac.uk/iar1/contact.html) please.

There are two main types of equation you need to know how to solve, namely linear equations and separable equations.

1. Linear Equations.

A first order linear differential equation looks like:

$$\frac{dx}{dt} + g(t)x = f(t) \qquad [1]$$

(That may be slightly different to how the lecturer wrote it, but is equivalent.) The standard way of solving this kind of equation consists of finding something called an integrating factor μ(t):

$$\mu(t) = \exp\left(\int g(t)dt\right) \qquad [2]$$

where exp(x) means e^x, ie "the exponential of".

The integral is an indefinite integral, but you do not need to worry about a constant of integration here.

So, for example, if we wish to solve the equation:

$$\frac{dx}{dt} + 3tx = t \qquad [3]$$

Then $g(t)$ is $3t$, and from [2] we would get:

$$\mu(t) = \exp\left(\int 3t\ dt\right) = \exp\left(3t^2/2\right) \qquad [4]$$

The point of $\mu(t)$ is that if you multiply the original equation through by $\mu(t)$, it makes the left hand side "exact", in other words, $\frac{d}{dt}$ of something. That "something" turns out to be $x\ \mu(t)$. In other words, taking [1] and multiplying through by $\mu(t)$:

$$\mu(t)\frac{dx}{dt} + \mu(t)g(t)x = \mu(t)f(t) \qquad [5]$$

which turns out to be $$\frac{d(x\,\mu(t))}{dt} = \mu(t)f(t) \qquad [6]$$

In our example, we'd get:

$$\exp\left(3t^2/2\right)\frac{dx}{dt} + 3tx\exp\left(3t^2/2\right) = t\exp\left(3t^2/2\right) \qquad [7]$$

which is

$$\frac{d\left(x\exp\left(3t^2/2\right)\right)}{dt} = t\exp\left(3t^2/2\right) \qquad [8]$$

and so, integrating both sides:

$$x\exp\left(3t^2/2\right) = \int t\exp\left(3t^2/2\right)dt = \frac{1}{3}\exp\left(3t^2/2\right) + k \qquad [9]$$

where k is a constant of integration. So the solution for x is:

$$x = \frac{1}{3} + k\exp\left(-3t^2/2\right) \qquad [10]$$

Try substituting this into [3] and check that it solves the equation.

There is another method of solving linear equations, which has the advantage that it also works for second and higher order equations, but the disadvantage that it requires $g(t)$ in [1] to be a constant. The method consists of finding any one solution to the equation (known as a *particular integral*, or a *particular solution*) and then adding to it any solution to the so-called *homogenous equation*, which one obtains by setting the right hand side of [1] equal to zero. This second part of the overall solution is known as the *complementary function*. This method may be covered in lectures in the very simple case of $f(t)$ in [1] being a constant, in which case the particular integral is just the steady-state solution to the equation, and the complementary function can be obtained via the method described below for separable equations.

2. Separable equations

A separable equation is one which can be written in the form:

$$g(x)\frac{dx}{dt} = f(t) \qquad [11]$$

It may not be presented to you in this form, but as long as you can manipulate your equation to get it into this form, your equation is separable.

The method used to solve equations in the form of [11] is very simple. We just separate the terms so that all the x's are on one side and all the t's are on the other:

$$g(x)dx = f(t)dt \qquad [12]$$

and then integrate both sides:

$$\int g(x)dx = \int f(t)dt \qquad [13]$$

So, for example, given the separable equation:

$$\frac{dx}{dt} = 3x^2 t \qquad [14]$$

which matches the form in [11] with $g(x) = x^{-2}$ and $f(t) = 3t$, we would get:

$$\int x^{-2} dx = \int 3t dt \qquad [15]$$

hence

$$-x^{-1} = 3t^2 / 2 + c \qquad [16]$$

where c is a constant. So:

$$x = \frac{-1}{3t^2 / 2 + c} \qquad [17]$$

The value of c can be found if we are given (say) the value of x at t=0.