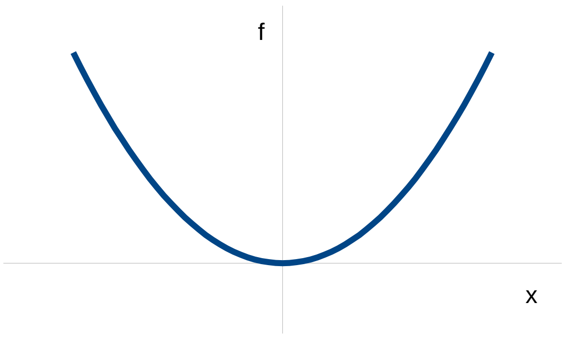
A Tutorial on the Definiteness of the Hessian and the Nature of Stationary Points

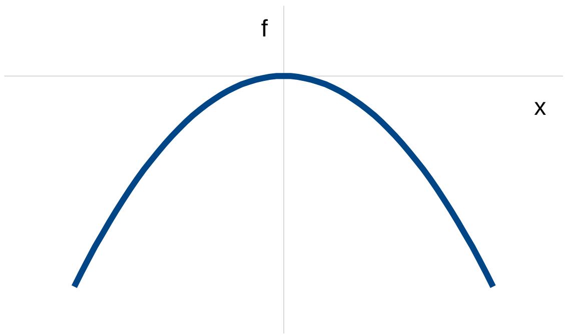
Corrections to Dr Ian Rudy (http://www.robinson.cam.ac.uk/iar1/contact.html) please.

This tutorial will help you to understand the link between the definiteness of the Hessian and the nature of a stationary point. We will start by analysing a function $f(x, y) = ax^2 + 2bxy + cy^2$ where a, b, c are parameters. This form of function is known as a *quadratic form*.

We need to visualise the function. Importantly, it is always zero when x and y are zero. It helps to start with some special cases:

(i) If $b = c = 0$, we can draw the function on a 2D diagram, with a convex parabola for $a > 0$ and a concave parabola for $a < 0$:

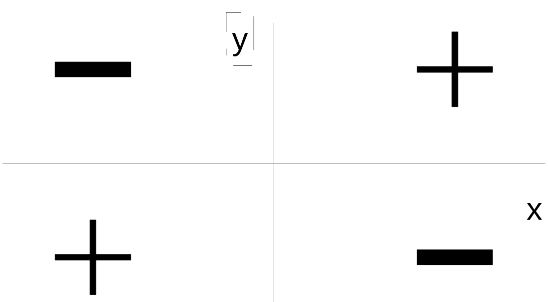




$a > 0$ $a < 0$

(ii) The case $a = b = 0$ will be similar to (i), except the role of the x-axis is replaced by the y-axis, and with c being the key parameter.

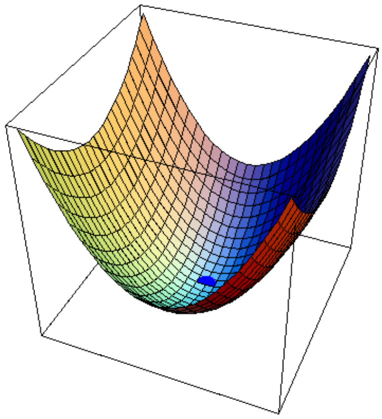
(iii) If a, c are negligible in comparison with the magnitude of b then the function changes sign as we move around the four quadrants of the xy plane. For $b > 0$ the sign of f would be as follows:



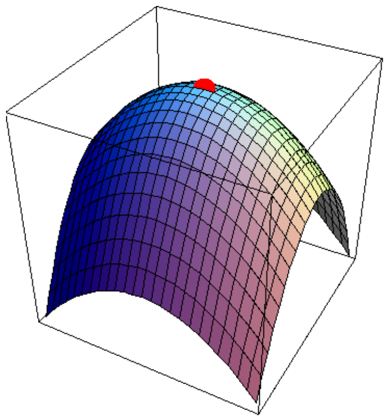
(iv) If a and c are of opposite sign, but large in comparison with the magnitude of b, then the function changes sign in a way that is similar to (iii), except with the plus and minus symbols rotated by 45 degrees.

Having thought about these special cases, you should now be able to understand qualitatively three-dimensional representations of the function. The graphs below are taken from http://mathinsight.org/local_extrema_introduction_two_variables. There are essentially three scenarios, which depend on the sign of $ac-b^2$:

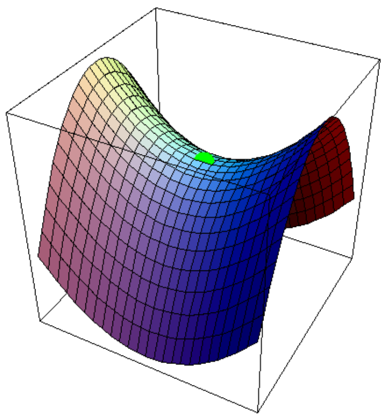
$ac-b^2>0, a>0$ (hence $c>0$)



$ac-b^2>0, a<0$ (hence $c<0$)



$ac-b^2<0$



The top graph results from special cases (i) and (ii) with $a>0, c>0$; the middle graph results from special cases (i) and (ii) with $a<0, c<0$; the bottom graph results from special case (iii) or (iv). Although it won't be obvious in detail why $ac-b^2$ is a key quantity, you should be able to understand its significance qualitatively from those special cases. If it isn't apparent to you why we can infer the sign of c from the sign of a when $ac-b^2>0$, note that b^2 must be positive, so if subtracting it from ac leaves a positive value, then a and c must have the same sign.

Now we can make a key observation. Remember that f is always zero at the origin. It's the blue, red, or green coloured blob respectively in the diagrams above. So the situation of f having a minimum at the origin is equivalent to saying $f>0$ for all $(x,y)\neq 0$. Similarly, the situation of f having a maximum at the origin is equivalent to saying $f<0$ for all $(x,y)\neq 0$.

Now let's find the Hessian of f . We get:

$$\boldsymbol{H}=\begin{pmatrix}2a & 2b\\ 2b & 2c\end{pmatrix}=2\begin{pmatrix}a & b\\ b & c\end{pmatrix}$$

But it also turns out that f can be written as $\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}a & b\\ b & c\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ (check this by expanding out this matrix expression). So $f=\frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\,\boldsymbol{x}$ where $\boldsymbol{x}=\begin{pmatrix}x\\ y\end{pmatrix}$.

So a statement that $f>0$ for all $(x,y)\neq 0$ is equivalent to a statement that $\boldsymbol{x}^T\boldsymbol{H}\,\boldsymbol{x}>0$ for all $\boldsymbol{x}\neq\boldsymbol{0}$. (The factor of $\frac{1}{2}$ can be dropped as it does not affect the sign.) But that is the definition of $\boldsymbol{H}$ being positive definite.

Hence, for $f(x,y)=ax^2+2bxy+cy^2$, if $a\neq 0$ and $c\neq 0$ and $ac-b^2\neq 0$:

f has a minimum at the origin $\Leftrightarrow$ $f>0$ for all $\boldsymbol{x}\neq 0$

$\Updownarrow$

H is positive definite at the origin $\Leftrightarrow$ $\boldsymbol{x}^T\boldsymbol{H}\,\boldsymbol{x}>0$ for all $\boldsymbol{x}\neq 0$

You can fill in obvious gap. We have established that f having a minimum at the origin is equivalent to $\boldsymbol{H}$ being positive definite at the origin. Similarly, f having a maximum at the origin is equivalent to $\boldsymbol{H}$ being negative definite there. If f has neither a maximum nor a minimum at the origin then $\boldsymbol{H}$ is indefinite there. The relatively rare cases involving $a=0$ and/or $c=0$ and/or $ac-b^2=0$ need more careful treatment, beyond the scope of this document.

We have established these properties for a rather specific form of function, namely $f(x,y)=ax^2+2bxy+cy^2$, a quadratic form. However it turns out that most continuous functions of two variables can be approximated by such a form close to a stationary point, if we redefine the origin to be at the stationary point. So the Hessian method is general. Again, the exceptions are when the implied values of a,b,c are such that $a=0$ and/or $c=0$ and/or $ac-b^2=0$.

Advanced and entirely optional: if you want to know more about the extension of this idea to continuous functions of two variables in general, look into the Taylor Series of two variables. The first order term in the Taylor Series will be zero because the first order partial derivatives are zero at a stationary point. And the zeroth order term will also be zero because we redefined the origin to be at the stationary point. Hence whether the function is positive or negative as we move away from the stationary point depends (in the first instance) on the second order term, which turns out to be our quadratic form.

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