ECONOMICS QUALIFYING EXAMINATION IN
ELEMENTARY MATHEMATICS

Wednesday 26 April 2000 1.30 to 4.30

This exam comprises ***two*** *sections. Each carries 50% of the total marks for the paper. You should attempt* ***all*** *questions from Section A and* ***two*** *questions from Section B.*

You are reminded that only the approved calculators may be used.

Graph paper and Mathematical Tables are provided

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SECTION A

1. (a) Find the derivatives dy/dx of:

   i. $y = \frac{(x^2-2x+3)^2}{4x}$

   ii. $y = \ln\left(\frac{x^2+1}{2x+3}\right)$

   (b) Find the partial derivative with respect to x of the function

$$f(x, y) = (x^3 - 2xy + y^2)(x - 2y)^2$$

2. (a) Let $C(q)$ represent a firm's total cost function, where q denotes the firm's output level. Show that at every turning point on the average cost function, average cost and marginal cost are equal

   (b) If $R(q)$ represents the firm's total revenue function, show that the equalisation of marginal revenue and marginal cost is a necessary, but not a sufficient, condition for profit maximisation. What is a sufficient condition?

3. Let

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix}, B = \begin{bmatrix} x \\ y \end{bmatrix}, C = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Find

   (a) AB
   (b) A^{-1}

   Hence solve the system of equations AB = C for x and y.

4. Minimise y $= x_1^2 + 2x_1x_2 + 4x_2^2$ subject to the constraint $x_1 + x_2 = 1$

(a) by substitution of the constraint, and

(b) by Lagrange's method.

5. (a) What are the maximum and minimum values of the function $f(x) = xe^{-x}$ for $x \geqslant 0$.

(b) Sketch the graph of this function.

6. A consumer's preferences are represented by the utility function:

$$u = x_1^{\frac{1}{3}}\ x_2^{\frac{2}{3}}$$

where u denotes utility and x_1 and x_2 denote the quantities. If the budget constraint is $m = p_1x_1 + p_2x_2$, determine the demand functions (that is, the optimal values of x_1 and x_2 in terms of p_1, p_2, and m) and show that the total expenditure on each commodity is not affected by the price of either.

7. (a) Find the following integrals

i. $\int 4e^{\frac{x}{4}}dx$

ii. $\int \frac{2x}{x^2+3}dx$

iii. $\int (x+3)\,(x+1)^{\frac{1}{2}}\,dx$

(b) Find the area under the following $f(x) = 5x^4 - 3x + 2$ between $x = -z$ and $x = z$

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8. Find an expression for q as a function of t, by solving the following difference equations

(a) $q_t - 2q_{t-1} = 0 \qquad q_0 = 1$

(b) $2q_t - q_{t-1} = 6 \qquad q_0 = 2$

9. Suppose that the quantities supplied and demanded for a commodity are given by:

$$q_t^s = 3p_{t-1} - 5$$

$$q_t^d = 25 - 2p_t$$

where q_t^s, q_t^d and p_t denote the quantity supplied, the quantity demanded and price at time t respectively.

(a) Find the equilibrium price p^* such that $p^* = p_t = p_{t-1}$.

(b) The explicit solution for p_t in terms of t, given that $p_o = 4$.

(c) Is p^* a stable equilibrium?

SECTION B

10 An economy produces output according to the production function

$$Q_t = AK_t^{\alpha}L_t^{\beta}$$

where Q_t is aggregate output at time t, K_t is the level of capital, L_t is labour and $A > 0$.

(a) Show that the conditions required for the marginal products of capital and labour to be positive and decreasing are that $0 < \alpha < 1$ and $0 < \beta < 1$.

(b) Show that the production function is homogeneous of degree $\alpha + \beta$.

(c) Euler's theorem states that if a function $f(x, y)$ is homogeneous degree λ then

$$\frac{df}{dx} \cdot x + \frac{df}{dy}.y = \lambda \ f \ (x, y).$$

Using Euler's theorem, show that the aggregate production function given in Equation (1) can only be used to represent a competitive economy, where the factors of production are paid their marginal products, if it also exhibits constant returns to scale.

(d) Assuming that the economy is competitive and the production function is given by Equation (1), show that output per worker can be written as an increasing and concave function of capital per worker, that the slope of this new function is the marginal product of capital, and that the share of total output assigned to the labour force is β

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11 Valva is a Dutch car manufacturer, producing left-hand drive cars for the market in continental Europe and right-hand drive cars for the UK market.

The demand for Valva cars on the continent is given by the equation

$$Q_C = 94,000 - 4p_C$$

and the demand for Valva cars in the UK by the equation

$$Q_{UK} = 60,000 - 2p_{UK}$$

where p_C is the continental price and p_{UK} is the UK price (both denominated in Euros).

Valva's production costs are not affected by the positioning of the driving mechanism, and given by the equation

$$C(Q) = Q^2, \qquad \text{where } Q = Q_{UK} + Q_C$$

(a) Write down Valva's total profits as a function of the number of cars supplied to each market.

(b) Determine the profit maximising production levels of right-hand drive and left-hand drive cars.

(c) What prices will Valva charge in each market?

(d) If UK legislation is passed requiring Valva to equalise the (euro denominated) price in both countries, what price will Valva cars sell at? What impact will this legislation have on continental consumers?

12 A model of aggregate demand in an economy takes the following form:

$$Y = C + I + G \tag{1}$$

$$C = Y - T - S \tag{2}$$

$$S = s(Y - T) \tag{3}$$

$$T = T_0 + tY \tag{4}$$

$$I = a - br \tag{5}$$

$$D = G - T \tag{6}$$

$$M^d = \alpha Y - \beta r \tag{7}$$

$$M^s = M_0 \tag{8}$$

where Y is national income, C is consumption, I is investment, S is savings, T is tax revenue, D is the government deficit, r is the interest rate, M^d is demand for money and M^s is the supply of money. The government is able to exogenously determine the values of G, T_0 and M_0. The parameters $a, b, s, t, \alpha, \beta$ are all positive, and both s and t are less than 1.

(a) Solve equations (1)-(5) for an expression to represent the IS curve. Show that r decrease with Y along the IS curve.

(b) Solve equations (1)-(8) for the equilibrium values of Y and r.

(c) What is the impact of an increase in M_0, the exogenous money supply, on the equilibrium values of Y?

(d) Identify a fiscal policy the government may adopt, which would increase the level of national income while reducing the government deficit. Justify your answer.

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13 A Cournot market consists of n identical firms which all individually select the quantity of a commodity to produce. Market demand for the commodity is given by

$$P = 100 - 5Q$$

where

$$Q = \sum_{i-1}^{n} q_i, \text{ and } q_i \text{ is the output of firm } i.$$

All firms posses an identical cost function, given by $c_i(q_i) = 2q_i$.

(a) Calculate the profit maximising output level for firm i, given that all other firms have selected to produce the same given quantity, $\overline{q}$.

(b) Determine the 'symmetric equilibrium' level of output, where all firms are maximising profits and producing the same quantities.

(c) What is the market price and the level of profits earned by each firm in equilibrium?

(d) Show that as the number of firms increases, the market becomes increasingly competitive, with prices approaching marginal cost and profits approaching zero.

14 A firm uses capital, k, and labour, l, according to the production function

$$4 \ln q = \ln k + \ln l$$

where q is the quantity produced.

The firm can hire labour at wage w, capital at a rental rate r, and faces demand for its product given by the function

$$q = 10 - 2p$$

(a) Show that the firm's cost function can be written $c(q) = 2(wr)^{1/2}q^2$.

(b) Determine the profit maximising level of production in terms of w and r

(c) How is the firm's demand for labour affected by changes in the wage rate?

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15 An individual gains utility from both consumption and leisure, represented by the utility function

$$u(x, l) = x^{1/4} l^{3/4}$$

where x is the level of consumption and l is the number of hours of leisure. The individual can work for as many hours as desired (up to a limit of 24 per day!) at the wage rate w, and let p denote the price of the consumption good.

(a) Write down the utility maximisation problem faced by the individual.

(b) Show that, irrespective of the wage rate, the individual will choose to work for 6 hours per day.

(c) If the government introduces a 10% sales tax on the consumption good, how much revenue will be generated, and what will the effect be on the individual's utility?

(d) An alternative policy would be for the government to raise the same revenue by imposing a lump sum tax instead of the sales tax. What would the effect be on hours worked?

END OF PAPER