ECONOMICS QUALIFYING EXAMINATION IN
ELEMENTARY MATHEMATICS

Wednesday 25 April 2001 1.30 to 4.30

This exam comprises ***two*** *sections. Each carries 50% of the total marks for the paper. You should attempt* ***all*** *questions from Section A and* ***two*** *questions from Section B.*

You are reminded that only the approved calculators may be used.

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SECTION A

1 (a) Find the derivatives dy/dx of:

i. $$y = \frac{x^2}{1+e^x}$$

ii. $$y = \frac{x^2+1}{x^2-1}$$

iii. $$y = x^3 \ln\left(x^3\right).$$

(b) If Q = $L^{0.5}$ $K^{0.5}$ show that:

$$Q = L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K}.$$

2 Given the Cobb-Douglas utility function:

$$u = 25x^\alpha y^\beta,$$

(a) Derive expressions for the marginal utilities with respect to x and y.

(b) Derive an expression for the slope of the indifference curve in terms of the marginal utilities.

(c) How does the slope of the indifference curve change as β changes?

3 A 5 year government bond pays the holder £100 after 5 years. It also pays an annual coupon of 8% in each of the 5 years. An individual holds 10 of these bonds. Assume that the market interest rate is 8% per year.

(a) Calculate the total value of the coupon payments paid each year (in £).

(b) Calculate the net present value of the coupon payments paid in each year (in £) and the net present value of the final payments made after 5 years (in £).

(c) Suppose that the market interest rate changes to 10% per year (but the coupon payments remain at 8%). Calculate the net present value of the coupon payments paid in each year (in £) and the net present value of the final payments made after 5 years (in £).

4 The demand function for a good is given by $Q + 1 = 200/P$. The supply function is given by $P = 5 + 0.5Q$.

(a) Rewrite the demand function so that it expresses P in terms of Q.

(b) Sketch the supply and demand functions.

(c) Calculate the equilibrium price and quantity.

5 A firm produces output Q from two inputs, K and L such that:

$$Q = 12L^{0.5}K^{0.5}.$$

Using Lagrange's method, find the values of L and K which minimise cost when the price of L is £25 per unit, the price of K is £30 per unit and no less than 240 units of output can be produced.

6 A firm has three products P_1, P_2 and P_3 which it sells to its two customers $C_1$and C_2. The number of items of each product sold to these customers is given by:

$$A = \begin{bmatrix} 6 & 7 & 9 \\ 2 & 1 & 2 \end{bmatrix}$$

where the i^{th} row refers to the i^{th} customer and the j^{th} column refers to the j^{th} product. The firm charges the same price to each customer given by:

$$B = [100\ 500\ 200]^T$$

where the j^{th}column refers to the j^{th} product and T refers to the transpose. The total cost to produce 1 unit of each good is:

$$C = \begin{bmatrix} 35 \\ 75 \\ 30 \end{bmatrix}$$

where the j^{th} row of C refers to the j^{th} product. In addition, let $E = [1 \quad 1]$. Calculate and interpret the following matrices:

(a) AB.

(b) AC.

(c) $EAB - EAC$.

7 (a) Find the following integrals:

i. $\int \sqrt{3x+4}\ dx$

ii. $\int 5\,(6x+3)^{-4}\ dx$

iii. $\int (1-3Q)^5\ dQ$

(b) Find the area under the function $y = x^2 - 1$ between $x = 0$ and $x = 1$.

8 Find an expression for y as a function of t by solving the following difference equations:

(a) $y_{t+1} = 0.8\,y_t, \quad y_1 = 5.$

(b) $y_{t+1} = 10 + 0.8\,y_{t,} \quad y_1 = 5.$

9 Suppose that investment is determined by:

$$I_t = 2.5\,(Y_t - Y_{t-1}), \quad Y_0 = 8.$$

Saving is determined by:

$$S_t = 0.1\,Y_t.$$

(a) Find an expression for equilibrium Y_t as a function of t.

(b) Is the equilibrium stable?

SECTION B

10 A firm uses labour, L and capital, K, to produce output, Q, according to the production function:

$$Q = L^{0.5} + K^{0.5}$$

The firm pays a wage rate w for labour and a rental rate r for capital.

(a) The firm wishes to minimise the cost $C = wL + rK$ of producing a fixed level of output. Show that the firm's demands for labour and capital are given by:

$$L = \frac{r^2}{(w+r)^2} Q^2 \text{ and } K = \frac{w^2}{(w+r)^2} Q^2.$$

Hence show that the firm's cost function is given by:

$$C(w, r, Q) = \frac{wr}{w+r} Q^2.$$

(b) If the firm is a perfect competitor in the product market and can sell its output at price $p = 60$, with $w = 1$ and $r = 3$, find the numerical value of the firm's profit-maximising level of output.

(c) Suppose that the firm can sell no more than 20 units of output at the price of 60. Write down the Lagrangean function for the firm's profit maximisation problem subject to the constraint on output sold. Calculate the value of the firm's profit-maximising output and the value of the Lagrange multiplier.

(d) Suppose that the firm can sell a small amount of additional units of output in the 'black' economy (but can sell no more than 20 units in the 'official' economy). What price would the firm be willing to accept for these additional 'black' economy units of output.

11 A consumer spends all of her income, Y, on consumption in periods 1 and 2, denoted C_1 and C_2 respectively. The consumer's preferences are given by the utility function $u(C_1, C_2)$. Any income not consumed in period 1 earns interest at rate r.

(a) Write down the Lagrangean function for the consumer's problem of maximising utility subject to the budget constraint.

(b) Derive the first-order conditions and characterise the solution to the consumer's problem.

(c) Assume that the utility function takes the form:

$$u(C_1, C_2) = C_1 - \frac{a}{2}{C_1}^2 + \beta(C_2 - \frac{a}{2}{C_2}^2), \qquad a > 0.$$

where β denotes the discount factor which equals $\frac{1}{1+r}$. Characterise the solution in this case.

(d) How would the solution to (c) change if $\beta > \frac{1}{1+r}$? Explain the economic intuition behind this result.

12 A monopolist manufacturer of digital cameras produces at constant marginal cost, c, and sells them to N identical retailers in Cambridge at price s per unit. The retailers sell the product to final consumers at no additional cost to themselves. The inverse demand function faced by the retailers is given by:

$$P(Q) = a - bQ, \qquad a > 0, \quad b > 0$$

where Q denotes the quantity demanded and p the price charged by the retailers.

(a) Each identical retailer maximises profits by choosing an output level, given the price charged by the monopolistic, $s\,(< a)$, and the output level of other retailers.

i. Write down the profit-maximising problem faced by retailer i.

ii. Show that the profit-maximising level of output for retailer i is:

$$q_i = \frac{a - b \sum_{1}^{N-1} q_j - s}{2b}$$

where $q_j (j \neq i)$ denotes the output level of other retailers.

iii. Assuming that the retailers are identical, solve for the equilibrium level of output for each retailer.

(b) Write down the monopolist's profit-maximising problem and solve for the profit-maximising value of s.

(c) Suppose that the N retailers form a cartel and jointly set Q to maximise profits. Find the profit maximising level of output for retailer i.

13 Consumption in time t is given by:

$$C(t) = \alpha + \beta_0 Y(t) + \beta_1 Y(t-1) + \beta_2 Y(t-2)$$

where $Y(t)$, $Y(t-1)$ and $Y(t-2)$ denote income in time $t, t-1$ and $t-2$ respectively; α, β_0, β_1 and β_2 are parameters.

(a) Find the immediate response of consumption in time $t, C(t)$, to a change in income in time $t, Y(t)$, in terms of the parameters.

(b) Given that income is in equilibrium when $Y(t) = Y(t-1) = Y(t-2) = \overline{Y}$, find the equilibrium value of consumption in terms of $\overline{Y}$ and the parameters.

(c) Suppose that income jumps from time t to $t+1$, from $Y(t) = \overline{Y}$ to $Y(t+1) = \overline{Y} + \Delta\overline{Y}$. Find the levels of consumption in periods $t+1, t+2, t+3$ and $t+4$ (that is, $C(t+1), C(t+2), C(t+3)$ and $C(t+4)$) in terms of $\overline{Y}, \Delta\overline{Y}$ and the parameters.

(d) Hence, find the long-run response of consumption to the change in income in terms of the parameters. Explain why the long-run response differs, in general, from the short-run response given in (a). For what values of the parameters are the two responses identical.

14 A firm has a total cost function $C = Q^2$ where C denotes the total cost and Q denotes the level of output. The firm's output can be sold in two distinct markets. In market A, the quantity demanded is given by:

$$Q_A = 80 - P_A.$$

And in market B, the quantity demanded is given by:

$$Q_B = 100 - P_B.$$

The prices in the two markets are denoted by P_A and P_B respectively.

(a) If the firm can act as a monopoly supplier in just one market, would it prefer to sell in market A or market B?

(b) Suppose that the firm can act as a monopoly supplier in both markets simultaneously (and that the prices in the two markets can differ). Write down an expression for total profits in the two markets as a function of the quantities sold Q_A and Q_B. Solve the firm's profit maximisation problem.

(c) Suppose that consumers in each market can buy from either market so that the prices in the two markets are identical. Write down an expression for the total quantity demanded and solve the firm's profit maximisation problem.

(d) Show that the maximised value of the firm's total profits in (b) and (c) are not equal. Explain the economic rationale for your result.

15 A consumer has the utility function:

$$U(x, y) = \alpha \log_e x + \beta \log_e y$$

where x denotes the consumption of good 1 and y denotes the consumption of good 2, with $0 < \alpha < 1$ and $\alpha + \beta = 1$. The prices of the two goods are p_x and p_y; the consumer's income is m. The consumer maximises utility subject to the budget constraint.

(a) Write down the consumer's problem and show that the demands for each good are:

$$x = \alpha m / p_x \quad \text{and} \quad y = \beta m / p_y.$$

(b) Write down an expression for maximised utility in terms of p_x, p_y and m.

(c) The government wishes to raise utility by subsidising the consumption of good 2. Assume that $p_x = £1$, $p_y = £4, m = £24, \alpha = 0.5$ and the government subsidy given to the consumer is £1 per unit. Find the total cost to the government of this scheme and the resulting increase in the consumer's utility.

(d) The government is considering a lump-sum cash payment as an alternative to the per unit subsidy. Write down a suitable optimisation problem for the government given that the new scheme has the same cost to the government as the old one. Which scheme does the consumer prefer. Explain the economic rationale for your result.