$\prod$ECONOMICS QUALIFYING EXAMINATION IN
ELEMENTARY MATHEMATICS

Wednesday 24th April 2002 1.30 to 4.30

*This exam comprises **two** sections. Each carries 50% of the total marks for the paper. You should attempt **all** questions from Section A and **two** questions from Section B.*

You are reminded that only the approved calculators may be used.

You may not start to read the questions printed on the subsequent pages of this questions paper until instructed that you may do so by the Invigilator

SECTION A

1 If

$$\begin{aligned} x &= t^3 - 2t \\ y &= \sqrt{t} \end{aligned}$$

and

$$f(x,y) = \left(x^2 + y^2\right)^{1/2}$$

Find $\partial f/\partial x, \partial f/\partial y$ and df/dt

2 Given the Cobb-Douglas utility function:

$$u = x^{\alpha} y^{1-\alpha},$$

(a) Derive expressions for the marginal utilities with respect to x and y.

(b) Find dy/dx along an indifference curve.

3 Let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and } C = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

Find

(a) AB
(b) A^{-1}
(c) CB
(d) BC

4 (a) What are the maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 - 9x + 4$$

in the interval $[-2, 3]$.

(b) Sketch the graph of this function on the interval $[-2, 3]$.

5 Minimise

$$y = -x_1^2 - 4x_1x_2 - 5$$

subject to the constraint $x_1 + 2x_2 = 5$

(a) By substitution of the constraint.
(b) By Lagrange's method.

6 (a) Show that the function

$$f(x,y) = x^3 + 3x^2y$$

is homogeneous of degree 3.

(b) Verify directly that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x,y)$$

7 Find the following integrals:

(a) $\int (3x-2)^2 \; dx$

(b) $\int e^{x+2} \; dx$

(c) $\int (u - 3/u)^2 \; du$

(d) $\int_{-2}^{2} (t^2 - 2t + 3\,)dt$

8 Given the function

$$F(x,y) = x + y/x$$

Find the derivative dy/dx along the isoquant $F = 2$.

9 A firm's workers work harder if their wages are higher, so that if the firm employs e workers at wage w its output is $f(\lambda(w)e)$ where $\lambda(w)$ is the effort function and $f(.)$ the production function. The firm is a price taker in the product market and wishes to choose wages and employment to maximise profits.

(a) Formulate the firm's profit maximisation problem.

(b) Show that the optimal choice of wages is where

$$\frac{w}{\lambda}\frac{d\lambda}{dw} = 1$$

SECTION B

10 A consumer has utility function $U(x_1, x_2) = x_1^{1/4} x_2^{3/4}$ where x_i $i = 1, 2$ denotes consumption of good i. The prices of the two goods are p_1 and p_2 and the consumer's income is equal to m. The consumer wishes to maximise her utility.

(a) Show using Lagrange's method that demands for each good are given by:

$$x_1 = \frac{1}{4}\left(\frac{m}{p_1}\right) \text{ and } x_2 = \frac{3}{4}\left(\frac{m}{p_2}\right)$$

(b) If $p_1 = 1, p_2 = 2$ and $m = 10$ what is the value of the consumer's maximised utility. If m rises to 11 what is the new value of the consumer's utility. What is the relation between the increase in utility and the value of the Lagrange multiplier in (a)?

(c) Suppose now with p_1, p_2 as in (b) and $m = 10$ that it also takes one unit of time to obtain a unit of x_1 and four units of time to obtain a unit of x_2. The consumer has 16 units of time available. Formulate this problem using Lagrange multipliers and obtain the optimal choices of x_1 and x_2.

(d) Calculate the value of the Lagrange multipliers in (c) and explain what conclusion you draw from the value of the Lagrange multiplier associated with the budget constraint $x_1 + 2x_2 = 10$.

11 A consumer has untility function $U(c, z) = \alpha \ln(c) + (1 - \alpha) \ln(z)$ where c is consumption and z leisure. The price of consumption is p and the consumer's income is $(24 - z)\, w$.

(a) Write down the Lagrangean function for the consumer's problem of maximising utility subject to the budget constraint.

(b) Derive the first-order conditions and characterise the solution to the consumer's problem.

(c) Suppose the government introduced a small proportionate income tax at rate t. What is the effect on labour supply?

(d) Suppose instead that the government decided to levy a lump sum tax T, how would this affect the optimal choice of labour supply?

(e) Explain why your answers in (c) and (d) differ.

12 The economy of the republic of Fredonia produces two goods, x_1 and x_2. The technology of production for these goods can be expressed in terms of a matrix A which takes the form

$$A = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$$

where the first column of A gives the inputs of x_1 and x_2 needed to produce an output of one unit of x_1, and the second column of A gives the inputs of x_1 and x_2 needed to produce an output of one unit of x_2.

(a) Compute the matrices $(I - A)$ and $(I - A)^{-1}$.

(b) Assume that there is no investment or government expenditure, and suppose that the population of Fredonia wish to consume the vector c given by

$$c^T = \begin{pmatrix} 10 & 5 \end{pmatrix}$$

Compute the value of $(I - A)^{-1}c$. What is the economic interpretation of this expression?

(c) Suppose that the requirements of labour per unit of output are given by the vector

$$l^T = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

so that it requires 2 units of labour to produce 1 unit of x_1 and 1 unit of labour to produce 1 unit of x_2. Compute the total amount of labour needed to produce the consumption vector c.

(d) Show that $(I - A)^{-1} = (I + A + A^2 + A^3 + A^4 + \ldots)$. What is the economic interpretation of this result?

13 Supply and demand in an economy at date t are given by

$$\begin{aligned} D_t &= 14 - 2P_t \\ S_t &= 2 + bP_t^e \end{aligned}$$

where P_t is the price at t, P_t^e is the expectation of the price level at t and b is a parameter measuring the responsiveness of supply to expected prices. Markets clear so that $S_t = D_t$.

To begin with assume that price expectations are formed as $P_t^e = P_{t-1}$.

(a) Derive the difference equation for the price level in the economy and solve for the equilibrium price P^* such that if $P_{t-1} = P^*$ then $P_t = P^*$.

(b) By writing the difference equation in the form $z_t = \phi z_{t-1}$ for an appropriate choice of z_t and ϕ solve for z_t in terms of z_0. Hence derive an expression for P_t in terms of P_0.

(c) What restriction is required for the equilibrium price to be stable?

(d) Suppose the expectations process changes so that $P_t^e = P_t$. What happens now?

14 Consider an economy with two electric power utilities, Notional Power and Powergone. Each utility can generate electricity at a constant marginal cost per unit of 30. The market demand curve for electricity is given by

$$P = 180 - Q$$

where P is the price and Q is the level of annual sales.

(a) Suppose that Notional Power aims to maximise profits assuming that the annual output of Powergone, Q_p, is fixed. Set out Notional Power's objective function, and derive Q_n, its optimal level of output, as a function of Q_p.

(b) Suppose that Powergone adopts a similar strategy in setting its output, and that the two utilities are identical. Derive the optimal level of output for each firm, and the market price of electricity.

(c) In order to increase competition in the electricity industry, the government compels each utility to split into two identical firms, and prohibits any cooperation between the firms. Assuming that all the firms continue to take the other firms' output as given, derive the total level of electricity output, and the price of electricity, in the new equilibrium.

(d) Calculate the change in the annual profits of all firms in the electricity industry which results from the government's intervention.

(e) Derive an expression for the price which will be charged if there are N identical firms in the industry, and show that as the number of firms becomes large this price tends to the value of marginal cost.

15 A small open economy is described by the following equations

Consumption	$C = 15 + 0.8Y$
Investment function	$I = 40 - 5i$
Government expenditures	$G = 40$
Exports	$X = 40e$
Imports	$M = 0.1Y$
Goods Market Equilibrium	$Y = C + I + G + X - M$
Money demand	$L = 85 + 0.2Y - 10i$
Money supply	$M/P = 55$
Money Market Equilibrium	$L = M/P$

where e is the exchange rate and prices are fixed at $P = 1$.

(a) Initially the exchange rate is fixed at $e = 1$. Derive the IS and LM curves in this case and hence calculate the equilibrium levels of output Y and the interest rate i.

(b) Calculate the trade balance.

(c) The government then allows the exchange rate to float. Assuming there is no capital account calculate the new equilibrium levels of Y and i and the equilibrium exchange rate e.

END OF PAPER