ECONOMICS QUALIFYING EXAMINATION IN ELEMENTARY MATHEMATICS

Wednesday 22 April 1998 1.30 to 4.30

This exam comprises two sections. Each carries 50% of the total marks for the paper. You should attempt **all** *questions from Section A and* **two** questions from Section B.

You are reminded that only the approved calculators may be used.

Graph paper and Mathematical Tables are provided.

SECTION A

1 (*a*) Find the derivatives dy/dx of:

(*i*) $y = x^2 e^x$

(*ii*) x^3

(*iii*) $\ln x$.

(*b*) Find the partial derivatives with respect to x_1, x_2 and x_3 of:

$$y = (x_1 + 4)(3x_1 + x_2)\, x_3 .$$

2 Suppose that a firm has the production function $q(k,l) = Ak^{\alpha}l^{1-\alpha}$ where $A > 0$ and

$0 < \alpha < 1$.

(*a*) Show that the marginal product of labour $\partial q(k,l)/\partial l$ is positive, if $k > 0$ and $l > 0$.

(*b*) Is the marginal product of labour increasing or decreasing with l (when k is fixed)?

3

⌊ - b 1 ⌋ ⌊ C ⌋ ⌊ a ⌋

where *Y*, *C*, *I* and *G* denote income, consumption, investment and government expenditure respectively; *a* and *b* are positive parameters.

Find (*a*) $\boldsymbol{P}\,\boldsymbol{Q}$
(*b*) $\boldsymbol{P}^{-1}$
(*c*) $\boldsymbol{P}^{-1}\boldsymbol{R}$.

4 (*a*) Minimise $y = 2x_1^2 + x_2 + 1$ with respect to x_1 and x_2 subject to the constraint $x_1 = m - x_2$ using Lagrange's method.

(*b*) What is the value of the Lagrange multiplier?

(*c*) What interpretation can you attribute to it?

5 (*a*) What are the maximum and minimum values of the function $f(x) = x^3 - 8x^2 + 16x - 1$ in the interval [0, 2]?

(*b*) Sketch the graph of this function in the interval [0, 2].

6 A consumer's preferences are represented by the utility function $u(x_1, x_2) = 2 \ln x_1 + \ln x_2$. If the budget constraint is $m = p_1x_1 + p_2x_2$, determine the demand functions (that is, the optimal values of x_1 and x_2 in terms of p_1, p_2, and m).

7 (*a*) Show that the function $z = f(x, y) = (x^2 + y^2)^{\frac{1}{2}} x^{\frac{1}{2}} y^{\frac{1}{2}}$ is homogeneous of degree 4.

(*b*) Verify by calculation that $\partial x \qquad \partial y$ as predicted by Euler's theorem.

8 Find the area under the curve $f(t) = 10t^4 + t + 1$ between $t = -1$ and $t = 1$.

$$q_t^s = 3\,p_{t-1} - 21500$$

$$q_t^d = 8500 - p_t$$

9 Suppose that the quantities supplied and demanded for a commodity are given by:

where $q_t^{\breve{}}$, $q_t^{\breve{}}$ and p_t denote the quantity supplied, the quantity demanded and price in time *t* respectively.

(*a*) Find the equilibrium price p^* such that $p = p_t = p_{t-1}$.

(*b*) Find the explicit solution for p_t in terms of *t* given that $p_0 = 7499$.

(*c*) Is p^* a stable equilibrium?

SECTION B

10 A firm uses labour, l, and capital, k, to produce output, q, according to the production function:

$$q = 8\, k^{\frac{1}{2}}\, l^{\frac{1}{4}} .$$

The firm pays a wage rate 5 for labour and a rental rate 4 for capital.

(*a*) Write down the firm's cost minimisation problem.

(*b*) Using Lagrange's method, show that the firm's cost function can be

$$c(q) = z\, q^{\frac{4}{3}}$$

written as:
where z is a positive constant.

(*c*) Suppose that the market price of output is fixed at p. Write down an expression for the firm's profit in terms of output.

(*d*) Solve the firm's profit maximisation problem and show that the firm's

$$q = \frac{27\ p^3}{64\ z^3} .$$

supply function is:

11 Suppose that the supply and demand functions of a commodity are given by:

$$q^s(p) = \frac{15}{2}p - 10 \qquad \text{(supply)}$$

$$q^d(p) = 40 - 5p \qquad \text{(demand).}$$

(*a*) Calculate the equilibrium price, p^*, and quantity, q^*.

(*b*) Suppose that the government imposes an excise tax of t per unit on suppliers. Calculate the new equilibrium price, p_t .

(*c*) Compare $(p_t - p)$ and t and comment.

(*d*) Suppose that the government imposes the tax on consumers rather than on suppliers. How does the relationship between $(p_t - p)$ and t change? Comment.

12 Consider an economy in which output is produced according to the following

$$Y_t = K^{\frac{1}{2}} L^{\frac{1}{2}}$$

production function:
where Y_t, K_t and L_t denote output, capital and labour in time t respectively.

(*a*) Write down an expression for output per worker, y_t, in terms of capital per worker, k_t.

(*b*) Sketch the function you derived in (*a*), $y_t = f(k_t)$. Show that the function is strictly concave.

Assume that net investment in time t, I_t, is determined by:

$$I_t = 0.3\, f(k_t) - 0.1\, k_t \qquad \text{and}$$

$$I_t = k_{t+1} - k_t \ .$$

(*c*) Calculate the level of capital per worker, k^*, where $k_t = k_{t+1}$.

(*d*) If the economy starts off with 4 units of capital per worker ($k_0 = 4$), does k_t converge to k^* as t increases?

13 A firm has a total cost function $c = q^2$, where c denotes cost and q denotes the level of output. There are two markets in which the firm's output can be sold. In market

$$q_A = 100 - p_A$$

A demand for the product is given by:
where p_A denotes the product price in market *A*. In market *B* demand for the product is

$$q_B = 120 - p_B$$

given by:
where P_B denotes the product price in market *B*.

(*a*) Suppose the firm can act as a monopoly supplier to one of the two markets. Which market will it supply to and how much output will it produce?

(*b*) Suppose the firm can act as a monopoly supplier to both markets. Write down an expression for the firm's total profits as a function of the quantities of output supplied to each market, q_A and q_B.

(*c*) Derive the firm's profit-maximising choices of q_A and q_B. What is the value of the firm's total profits?

(*d*) Suppose consumers in each market can, at no extra cost, purchase the firm's product from the other market. Then there is effectively a single market with a single equilibrium price. Write down an expression for total product demand in the market, q, as a function of price, p. Derive the firm's profit-maximising choice of output in this case. What is the equilibrium price and the value of firm profits?

(*e*) Comment briefly on your results.

14 A model of aggregate demand in an economy takes the following form:

$$Y = C + I + G \quad (1)$$

$$C = b(Y - T) \quad (2)$$

$$I = a - hr \quad (3)$$

$$G = T \quad (4)$$

$$M_d = kY - jr \quad (5)$$

$$Ms = \bar{M} \quad (6)$$

where Y denotes national income, C denotes consumption, I denotes investment, G denotes government expenditure, T denotes taxes, r denotes the interest rate, M_d denotes the demand for money, and M_s denotes the supply of money. The government determines the money supply exogeneously at the value $\bar{M}$. The government also determines the level of government expenditure, but is required to run a balanced budget, as equation (4) shows. The parameters of the model, all of which are positive, are a, b, h, k, and j; $b < 1$.

(*a*) Solve equations (5) and (6) for an expression to represent the LM curve. Show that r increases with Y along the LM curve.

(*b*) Solve equations (1)-(4) for an expression to represent the IS curve. Show that r decreases with Y along the IS curve.

(*c*) Solve for the equilibrium values of Y and r.

(*d*) What happens to the equilibrium values of Y and r if the government increases the money supply?

(*e*) Show that an increase in government expenditure increases the equilibrium value of Y by less than the increase in G. Under what conditions would the increase in the equilibrium value of Y be equal to the increase in G? Given an economic interpretation of you anwer.

15 A monopolist manufacturer of video recorders produces at constant marginal cost, c, and sells them to N identical retailers, indexed by $i = 1,...,N$, in Cambridge at price s per unit. The retailers sell the product to final consumers at no additional cost to themselves. The inverse demand function faced by retailers is given by:

$$P(Q) = a - bQ \quad a > 0 \quad b > 0 \ .$$

Where Q denotes the quantity demanded and P the retailers' price.

Each identical retailier maximises profits by choosing an output level, given the price charged by the monopolist, $s(< a)$, and the output level of other retailers.

(a) Write down the profit maximisation problem faced by retailer i.

(b) Show that the profit-maximising level of output for retailer i is decreasing in the output of other retailiers.

(c) Using the assumption that the retailers are identical, solve for the equilibrium level of output for each retailer.

(d) Write down the monopolist's profit-maximisation problem and solve for the profit-maximising value of s.