Section A

1. *(a)* Find the derivatives dy/dx of:

$$y = \ln\left(x^2\right) + e^{-x}$$

$$y = \left(4x - x^2\right)^{\frac{1}{2}} (x+3)^{-\frac{3}{2}}$$

   (b) Find the partial derivative with respect to x and y of the function

$$z = f(x, y) = \frac{\left(x^3 + y^{-\frac{1}{2}} - 3\right)^{\frac{1}{2}}}{x} y^{\frac{3}{2}}$$

2. *(a)* Suppose that a firm has a production function

$$q(K, L) = AK^{\beta}L^{1-\beta} \qquad 1 > \beta > 0, \quad A > 0$$

   Show that the marginal product of capital, $\partial q/\partial K$, is positive if K and L are positive.

   (b) How does an increase in the quantity of labour supplied affect the marginal product of capital?

   (c) If $\beta = \frac{1}{2}$, and $A = 1$, write down an expression for output per worker in terms of capital per worker.

3. Let

$$A = \begin{bmatrix} 1 & -\beta \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} c \\ y \end{bmatrix} \text{ and } R = \begin{bmatrix} \alpha \\ I + G \end{bmatrix}$$

   where c, y, I and G denote consumption, income, investment and government spending, respectively. α and β are both positive parameters.

   Find:

   (a) AB
   (b) A^{-1}

   Hence solve the system of equations $AB = R$ for y and c.

4. Minimise

$$y = \frac{1}{4}x_1 + \frac{1}{2}x_2^2$$

subject to the constraint: $x_1 + x_2 = 1$

(a) by substitution of the constraint

(b) by Lagrange's Method

5. *(a)* What are the maximum and minimum values of the function

$$f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$$

in the interval $[1, 7]$.

(b) Sketch the graph of this function in the interval $[1, 7]$

6. A consumer's preferences are represented by the utility function

$$u = x_1^{\frac{1}{2}} + 2x_2^{\frac{1}{2}}$$

where u denotes utility and x_1 and x_2 are the quantities consumed. Given prices p_1 and p_2 and total income of m, find the demand functions for the two goods.

7. A monopolist has total costs of producing q of

$$c = \frac{1}{2}q^2 + 4q + 1$$

where c denotes cost and q denotes output. Demand for the product is given by

$$q = 22 - p$$

where p is the price of the output.

(a) Write down an expression for the monopolist's profits as a function of q.

(b) How much will the monopolist choose to produce and what will be the equilibrium price for the product?

8. *(a)* Evaluate the following integrals:

$$\int_0^3 \left(1 + x^2\right) dx$$
$$\int_{-1}^1 2e^{-x} dx$$

(b) Find an expression for p as a function of t by solving the following difference equations:

$$\begin{array}{lll} \textit{(i)} & : & p_t = 0.25p_{t-1}, \qquad p_0 = 1 \\ \textit{(ii)} & : & p_t = 0.5p_{t-1} + 8, \qquad p_1 = 1 \end{array}$$

9. Demand and supply in a particular market are given by

$$\begin{aligned} q_t^d &= 400 - p_t \\ q_t^s &= \alpha p_{t-1} - 800 \end{aligned}$$

where q_t^d, q_t^s and p_t denote the quantity demanded, quantity supplied and the price in time t, respectively.

(a) Find the equilibrium price, p^*, such that if $p_{t-1} = p^*$, then $p_t = p^*$.

(b) For what values of α is p^* a stable equilibrium.

Section B

10. An individual gets utility from consumption, c, and from leisure, z, according to the utility function

$$u(c, z) = c^{\alpha} z^{1-\alpha}$$

There are 24 hours available each day, and time not spent as leisure is spent working for a wage, w. The individual receives no other income. The price of consumption is p $0 < \alpha < 1$.

(a) Assuming that all income is spent on consumption, show using Lagrange's method, that

$$\begin{aligned} z &= (1-\alpha) 24 \\ c &= \alpha 24 \frac{w}{p} \end{aligned}$$

(b) If $w = 10$, $p = 2$, and $\alpha = 0.5$, what is the maximised value of utility.

(c) The government now introduces a 20% tax on wage income. What happens to the number of hours worked and the level of utility achieved? How much revenue is raised?

(d) Instead of the income tax, the government imposes a lumpsum tax which raises the same amount of revenue as the income tax in part (c). How many hours are worked now? Compare the level of utility under the lumpsum tax with the utility under the income tax.

11. A firm uses labour l and capital k to produce output q according to the production function

$$q(k,l) = k^{\beta} l^{1-\beta}$$

The firm pays a wage of w and a rental rate of r.

(a) Write down the firm's cost minimsation problem and solve to show that the demand for labour and demand for capital is given by

$$l = \bar{q}\left(\frac{r}{w}\frac{1-\beta}{\beta}\right)^{\beta}$$

$$k = \bar{q}\left(\frac{w}{r}\frac{\beta}{1-\beta}\right)^{1-\beta}$$

(b) Suppose a retailer wants to purchase a quantity $\bar{q} = 50$ of the firm's output and suppose $w = 4$, $r = 16$ and $\beta = 0.5$. What is the minimum price that the retailer has to offer to supply $\bar{q}$? How much capital and labour will the firm use to produce $\bar{q}$?

(c) Suppose the firm is constrained to use 20 units of capital. Using Lagrange's method, find the firm's demand for labour and minimised cost of producing $\bar{q}$ and hence the minimum price that the retailer has to offer.

(d) Suppose that the firm can obtain an extra amount of capital by paying over the going rate of $r = 16$. What is maximum premium, approximately, that the firm would be willing to pay? [Think of Lagrange multiplier in (c)]

12. The balance of trade is given by

$$B = eX(e) - M(e, y)$$

where B is the balance of trade, e is the foreign price of domestic currency, $X(e)$ is the value of exports (a function of e) and $M(e, y)$ is the value of imports, as a function of e and y.

(a) What is the effect of a change in e on the balance of trade?

(b) What is the effect of a change in y on the balance of trade?

(c) By how much does the exchange rate have to adjust following an increase in y in order to keep the trade balance equal to zero. (i.e. $dB = 0$)

(d) Discuss the effect of a change in government spending on the exchange rate.

13. Aggregate demand is described as follows:

$$\begin{aligned} y &= c + I + G && (1) \\ c &= \beta (y - T) && (2) \\ I &= \alpha - \gamma r && (3) \\ T &= \bar{T} + ty && (4) \\ M^d &= \delta(Y - T) - \varepsilon r && (5) \\ M^s &= \bar{M} && (6) \end{aligned}$$

where c is consumption, y is income, I investment, r the interest rate, M^d demand for money, M^s money supply, G government spending, T tax revenue, $\bar{T}$ a lumpsum tax and t the proportional tax on income. Greek letters are parameters. The government has control over G, $\bar{T}$, t and $\bar{M}$.

(a) Derive an expression for equilibrium in the goods market (the IS curve). Show that r decreases with Y along the IS curve.

(b) Derive an expression for equilibrium in the money market (the LM curve). Show that r increases with Y along the LM curve.

(c) Solve for equilibrium values of y and r.

(d) What happens to y and r if the government increases G?

(e) As the government increases government spending, G, it increases $\bar{T}$ to pay for the increased expenditure, holding t fixed. By how much does $\bar{T}$ have to increase to fund the change in G, dG? (i.e. such that $\frac{d(G-\bar{T}-ty)}{dG} = 0$)

14. A firm has total cost function $c = q^2$. There are two countries in which the firm's output can be sold (and a common currency between the two countries): in country A, demand is given by

$$q_A = 12 - p_A$$

In country B, demand is given by

$$q_B = 16 - p_B$$

where p^A and p^B are the prices in country A and B respectively.

(a) If the firm can choose one country in which it will be the monopoly supplier, which country will it choose?

(b) Suppose the firm can act as a monopolist in both countries and there is no cross-border shopping between the two countries. Write down an expression for total profits and derive the firm's profit maximising choices of q_A and q_B. What is the value of the firm's profits?

(c) Suppose there is no longer segmentation and consumers can purchase the good from either country. Write down an expression for total product demand as a function of price. What is the firm's profit maximising choice of output? What is the equilibrium price level?

15. A monopolist manufacturer of buses produces at constant marginal cost, c and sells to N identical bus retailers, indexed by $i = 1, ..., N$ at price s per unit. The retailers sell the buses at no extra cost. The inverse demand function faced by retailers is given by

$$p(q) = \alpha - \beta q \qquad \alpha > 0, \beta > 0$$

Each identical retailer maximises profits by choosing output, taking the price charged by the monopolist, $s\,(< \alpha)$, and the quantity of other retailers as given.

(a) Write down the profit maximisation problem faced by retailer i.

(b) Show that the profit maximising level of output of i is given by

$$q_i = \frac{\alpha - \beta \sum_{j \neq i}^{N} q_j - s}{2\beta}$$

and show that the output of firm i is decreasing in the output of other retailers.

(c) What is the equilibrium value of output for each retailer (assuming retailers are identical)?

(d) What is the profit-maximising value of s, the price set by the monopolist?