ECONOMICS QUALIFYING EXAMINATION IN ELEMENTARY MATHEMATICS

Friday 28th September 2001 9 to 12

This exam comprises ***two*** *sections. Each carries 50% of the total marks for the paper. You should attempt* ***all*** *questions from Section A and* ***two*** *questions from Section B.*

You are reminded that only the approved calculators may be used.

You may not start to read the questions printed on the subsequent pages of this questions paper until instructed that you may do so by the Invigilator

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SECTION A

1 (a) Find the derivatives dy/dx of:

i. $$y = \frac{e^x}{1+e^x}$$

ii. $$y = \frac{5x^2}{(1+x)^2}$$

iii. $$y = \ln\left(x^3\right).$$

(b) Find the partial derivatives with respect to u and v of:

$$y = (u^2+3)(4u-v)$$

2 Given the Cobb-Douglas utility function:

$$u = x^\alpha y^{1-\alpha},$$

(a) Derive expressions for the marginal utilities with respect to x and y.

(b) Derive an expression for the slope of the indifference curve in terms of the marginal utilities.

(c) How does the slope of the indifference curve change as α changes?

3 Let

$$A = \begin{bmatrix} 1 & -b \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} c \\ y \end{bmatrix} \quad \text{and } R = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

where $0 < b < 1$. Find

$$\begin{array}{ll} (a) & AB \\ (b) & A^{-1} \end{array}$$

Hence solve the system $AB = R$ for c and y.

4 (a) What are the maximum and minimum values of the function

$$f(x) = -\frac{2}{3}x^3 + x^2 + 4x + 2$$

in the interval $[-2, 3]$.

(b) Sketch the graph of this function on the interval $[-2, 3]$.

5 Maximise

$$y = -x_1^2 - 3x_2 - 5$$

subject to the constraint $3x_1 + 2x_2 = 5$

(a) By substitution of the constraint
(b) By Lagrange's method

6 (a) Show that the function

$$f(x,y) = (x^2 + xy)^{\frac{1}{2}} y$$

is homogeneous of degree 2.

(b) Verify directly that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f(x,y)$$

7 (a) Find the following integrals:

i. $\int (x - 2x^2)\ dx$

ii. $\int e^{-2x}\, dx$

iii. $\int (1 - 3Q)^2\ dQ$

(b) Find the area under the function $f(t) = t^3 - 3t^2$ between $t = -1$ and $t = 1$.

8 Given the function

$$F(x,y) = x^2 y + 3xy^3$$

Find the derivative $\partial y/\partial x$ along the isoquant $F = 2$.

9 Demand and supply in a particular market are given by

$$\begin{aligned} Q_t^D &= 40 - P_t \\ Q_t^S &= \phi P_{t-1} + 20 \end{aligned}$$

where Q_t^D, Q_t^S, P_t denote the quantity demanded, quantity supplied and price at time t respectively

(a) Find the equilibrium price P^* such that if $P_{t-1} = P^*$ then $P_t = P^*$

(b) For what values of ϕ is the equilibrium stable?

SECTION B

10 A firm uses labour, L and capital, K, to produce output, Q, according to the production function:

$$Q = 4L^{0.5}K^{0.5}$$

The firm pays a wage rate w for labour and a rental rate r for capital.

(a) Suppose the firm wishes to minimise the cost $C = wL + rK$ of producing a fixed level of output. Show that the firm's demands for labour and capital are given by:

$$L = \frac{1}{4}\left(\frac{r}{w}\right)^{0.5} Q \text{ and } K = \frac{1}{4}\left(\frac{w}{r}\right)^{0.5} Q.$$

Hence show that the firm's cost function is given by:

$$C(w, r, Q) = \frac{Q}{2}(wr)^{0.5}.$$

(b) Suppose the firm is a perfect competitor in the product market and can sell its output at price $p = 3$. If the rental price of capital $r = 9$, what is the maximum wage rate the firm can pay without making a loss?

(c) Suppose a retailer wants to purchase a quantity $\overline{Q} = 120$ of the firm's output. If $w = 1, r = 9$ what is the minimum price per unit of output, p, that the retailer has to offer to induce the firm to supply $\overline{Q}$? How much capital and labour will the firm use to produce $\overline{Q}$?.

(d) Suppose that the firm is constrained to use a fixed amount of capital $\overline{K}$. Using the Lagrangean technique obtain the firm's demand for labour and minimised cost of producing output $\overline{Q}$.

11 A consumer lives for two periods 1 and 2 and earns income Y_1 and Y_2 and consumes C_1 and C_2 in each period respectively. The consumer's preferences are given by the utility function

$$u(C_1, C_2) = \ln(C_1) + \beta \ln(C_2)$$

The consumer can lend or borrow from period 1 to 2 at interest rate r.

(a) Write down the Lagrangean function for the consumer's problem of maximising utility subject to the budget constraint.

(b) Derive the first-order conditions and characterise the solution to the consumer's problem.

(c) What is the effect on C_1 of a change in β, a change in r and a change in Y_1?

(d) How would the solution to (c) change if the consumer was unable to borrow or lend in period 1?

12 An airline is a monopolist on a particular route such that if it charges a price P per seat the number of seats demanded is given by

$$Q = 130 - 3P$$

and the cost of supplying Q seats is $C(Q) = 10Q$

(a) Write down an expression for the airlines profit as a function of the quantity sold and solve the airline's profit maximisation problem?

(b) The airline then realises that there are two identifiable types of customers, business class and economy labelled A and B respectively with demand curves

$$\begin{aligned} Q_A &= 80 - P_A \\ Q_B &= 50 - 2P_B \end{aligned}$$

If airline tickets cannot be transferred between customers calculate the profit maximising behaviour for the airline in this case.

(c) Show that the airline's total profits in (a) and (b) differ and explain the economic rationale for your result.

13 An individual has utility function defined over consumption c and leisure z :

$$U(c, z) = c^{\frac{1}{2}} + z^{\frac{1}{2}}$$

There are 24 hours in a day and the time not spent as leisure is spent working for a wage w. The individual receives no other income and all income is spent on consumption at price p

(a) Write down the maximisation problem of the individual and show that it implies:

$$c = 24\left(\frac{w^2}{wp + p^2}\right) \quad \text{and} \quad z = 24\left(\frac{p^2}{wp + p^2}\right)$$

(b) Write down an expression for maximised utility.

(c) The government is considering introducing either a small tax on consumption at rate v or a small tax on income at rate t. Find and compare the effect on labour supply of each of these two policies.

14 Consider the following model of an economy. Output at time t, Y_t, is given by

$$Y_t = C_t + I_t + G_t$$

where consumption $C_t = bY_t$ with b the marginal propensity to consume $(0 < b < 1)$ and G_t represents the level of government spending. The capital stock of the economy is a fixed fraction of the level of output $K_t = \theta Y_t$.

(a) Using the fact that investment equals the change in the capital stock $I_t = K_t - K_{t-1}$ show that output in this economy follows the difference equation

$$Y_t = \frac{-\theta Y_{t-1}}{1 - b - \theta} + \frac{G_t}{1 - b - \theta}$$

(b) Find the immediate response of output at time t, Y_t, to a change in government spending at time t, G_t

(c) If government spending is fixed at $\overline{G}$ find the equilibrium values of output, consumption and investment.

(d) Find the long-run response of output to a change in government spending in terms of the parameters. Explain why the long-run response differs from the short-run response given in (b).

(e) Discuss what happens to this economy if $\theta = 0.3$ and $b = 0.6$.

15 An economy is described by the following equations

Savings function	$S = -15 + 0.1Y_D$
Investment function	$I = 40 - 5i$
Government expenditures	$G = 50$
Flat Taxes	$T = 50$
Disposable Income	$Y_D = Y - T$
Goods Market Equilibrium	$S + T = I + G$
Money demand	$L = 85 + 0.2Y - 10i$
Money supply	$M/P = 75$
Money Market Equilibrium	$L = M/P$

and prices are fixed at $P = 1$.

(a) Derive the IS and LM curves and hence calculate the equilibrium levels of output Y and interest rates i.

(b) The full employment level of output in this economy is 325. There are various fiscal and monetary actions that government may take to move the economy to full employment:

(i) If the government seeks to reach full employment by changing only the level of government spending G calculate the value of the government spending multiplier in this model and hence the level to which G must be set to bring about full employment. What is the size of the government's deficit or surplus?

(ii) If the government seeks to reach full employment by changing only the level of tax receipts T calculate the level to which T must be set to bring about full employment. What is the size of the government's deficit or surplus in this case?

(iii) Calculate the equilibrium levels of interest rates in (i) and (ii).

(iv) It is also possible for the government to move the economy to full employment by reliance on monetary policy alone. Calculate the level to which M must be set to bring about full employment. What is the level of interest rates in this case? If the government seeks to achieve full employment while maintaining the interest rate unchanged at its original level calculate the combination of fiscal policy (G) and monetary policy (M) required.