ECONOMICS QUALIFYING EXAMINATION IN ELEMENTARY MATHEMATICS

Friday 2 October 1998 9 to 12

This exam comprises two sections. Each carries 50% of the total marks for the paper. You should attempt **all** *questions from Section A and* **two** questions from Section B.
You are reminded that only the approved calculators may be used.
Graph paper and Mathematical Tables are provided.

SECTION A

1 (*a*) Find the derivatives dy/dx of:

(*i*) $y = x^2 e^x$

(*ii*) $y = (2x - 1)^{\frac{3}{2}} (x - 4)^{-\frac{1}{2}}$

(*b*) Find the partial derivative with respect to x of the function $z = f(x, y) = (x^2 + y^2)^{\frac{3}{2}} x^{\frac{1}{2}} y^{\frac{1}{2}}$

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2 (a) Suppose that a firm has the production function $q(k,l) = A\,k^{\alpha}\,l^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. Show that the marginal product of labour $\partial q(k,l)/\partial l$ is positive, if $k > 0$ and $l > 0$.

(b) Consider an economy in which output is produced according to the following production function:

$$Y_t = K_t^{\frac{1}{2}} L_t^{\frac{1}{2}}$$

where Y_t, K_t and L_t denote output, capital and labour in time t respectively. Write down an expression for output per worker, y_t, in terms of capital per worker, k_t.

3 Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} Y \\ C \end{bmatrix}$ and $\boldsymbol{R} = \begin{bmatrix} I + G \\ a \end{bmatrix}$

where Y, C, I and G denote income, consumption, investment and government
expenditure respectively; a and b are positive parameters.

Find (*a*) $\boldsymbol{AB}$
(*b*) $\boldsymbol{A}^{-1}$
(*c*) $\boldsymbol{A}^{-1}\boldsymbol{R}$.

4 Let

$$x + ay = 3$$
$$2x + 2y = 4$$

(*a*) Write these equations in matrix form $\boldsymbol{A}\,\underset{\sim}{x} = \underset{\sim}{b}$

(*b*) What happens to the determinant of $\boldsymbol{A}$ when $a = 1$?

(*c*) Does a solution to the equations for x and y exist when $a = 1$?

5 Minimise $y = x_1^2 + 2x_2^2$ subject to the constraint $x_1 + x_2 = 1$

(*a*) by substitution of the constraint, and

(*b*) by Lagrange's method.

6 (*a*) What are the maximum and minimum values of the function $f(x) = x^3 - 8x^2 + 16x - 1$ in the interval [0, 4]?

(*b*) Sketch the graph of this function in the interval [0, 4].

7 A consumer's preferences are represented by the utility function:

$$u = x_1^2 + x_2$$

where u denotes utility and x_1 and x_2 denote the quantities. If the budget constraint is $m = p_1x_1 + p_2x_2$, determine the demand functions (that is, the optimal values of x_1 and x_2 in terms of p_1, p_2, and m).

8 Find the area under the curve $f(t) = 10t^3 + t + 1$ between $t = -1$ and $t = 1$.

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9 Suppose that the quantities supplied and demanded for a commodity are given by:

$$q_t^s = 3p_{t-1} - 21500$$

$$q_t^d = 8500 - p_t$$

where q_t^s, q_t^d and p_t denote the quantity supplied, the quantity demanded and price in time t respectively.

(*a*) Find the equilibrium price p^* such that $p^* = p_t = p_{t-1}$.
(*b*) Is p^* a stable equilibrium?

SECTION B

10 A consumer has utility function $U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ where $x_i (i=1,2)$ denotes the consumption of good *I* and $0 < \alpha < 1$. The prices of the two goods are p_1 and p_2 and the consumer's income is equal to m .

(*a*) Assuming the consumer maximises utility subject to her budget constraint, show , using Lagrange's method, that her demands for each good are given by

$$x_1 = \frac{\alpha m}{p_1}, \qquad x_2 = \frac{(1-\alpha)m}{p_2}$$

(*b*) If $p_1 = 1,\ p_2 = 4,\ m = 24$ and $\alpha = \frac{1}{2}$, what is the value of the consumer's maximised utility?

(*c*) Suppose *m* rises to 25. What is the new value of the consumer's maximised utility. What is the relation between the increase in utility and the value of the Lagrange multiplier in (*a*)?

(d) Suppose it also takes 2 units of time to obtain one unit of x_1 and 1 unit of time to obtain one unit of x_2. The consumer has 15 units of time available. With m = 25 formulate this problem using Lagrange multipliers and solve for the optimal choices of x_1 and x_2. What interpretation can you give to the ratio of the Lagrange multipliers in this problem.

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11 Let $B = e\,X(e) - M(e)$

where B = balance of trade

e = foreign price of domestic currency

X = value of exports, X = X(e)

M = value of imports, M = M (e)

(*a*) Derive $\frac{dB}{de}$

(*b*) Use your result from (*a*) to derive the Marshall-Lerner condition.

(*c*) Comment on the assumptions behind the derivation of the Marshall-Lerner condition.

12 A firm uses labour, l, and capital, k, to produce output, q, according to the production function:

$$q = 8k^{\frac{1}{2}}l^{\frac{1}{4}}.$$

The firm pays a wage rate 5 for labour and a rental rate 4 for capital.

(*a*) Write down the firm's cost minimisation problem.

(*b*) Using Lagrange's method, show that the firm's cost function can be written as:

$$c(q) = zq^{\frac{4}{3}}$$

where z is a positive constant.

(*c*) Suppose that the market price of output is fixed at p. Write down an expression for the firm's profit in terms of output.

(*d*) Solve the firm's profit maximisation problem and show that the firm's supply function is:

$$q = \frac{27p^3}{64z^3}.$$

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13 A firm has a total cost function $c = q^2$, where c denotes cost and q denotes the level of output. There are two markets in which the firm's output can be sold. In market A demand for the product is given by:

$$q_A = 100 - p_A$$

where p_A denotes the product price in market A. In market B demand for the product is given by:

$$q_B = 120 - p_B$$

where P_B denotes the product price in market B.

(*a*) Suppose the firm can act as a monopoly supplier to one of the two markets. Which market will it supply to and how much output will it produce?

(*b*) Suppose the firm can act as a monopoly supplier to both markets. Write down an expression for the firm's total profits as a function of the quantities of output supplied to each market, q_A and q_B.

(*c*) Derive the firm's profit-maximising choices of q_A and q_B. What is the value of the firm's total profits?

(*d*) Suppose consumers in each market can, at no extra cost, purchase the firm's product from the other market. Then there is effectively a single market with a single equilibrium price. Write down an expression for total product demand in the market, q, as a function of price, p. Derive the firm's profit-maximising choice of output in this case. What is the equilibrium price and the value of firm profits?

(*e*) Comment briefly on your results.

14 A model of aggregate demand in an economy takes the following form:

$$Y = C + I + G \quad (1)$$
$$C = b(Y - T) \quad (2)$$
$$I = a - hr \quad (3)$$
$$T = T_0 + tY \quad (4)$$
$$M_d = kY - jr \quad (5)$$
$$M_s = \overline{M} \quad (6)$$

where Y denotes national income, C denotes consumption, I denotes investment, G denotes government expenditure, T denotes tax revenue, r denotes the interest rate, M_d denotes the demand for money, and M_s denotes the supply of money. The government determines the money supply exogenously at the value $\overline{M}$. The government also determines the level of government expenditure exogenously. T_0 denotes exogenous taxes. The parameters of the model, all of which are positive, are *a, b, h, k, t* and j; $b < 1$.

(*a*) Solve equations (1)-(4) for an expression to represent the IS curve. Show that r decreases with Y along the IS curve.

(*b*) Solve equations (1)-(6) for the equilibrium values of Y and r.

(c) What happens to the equilibrium value of Y if the government increases T_o?

(*d*) Derive the balanced budget multiplier, where $dG=dT_0$.

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15 A monopolist manufacturer of video recorders produces at constant marginal cost, c, and sells them to N identical retailers, indexed by $I = 1,...,N$, in Cambridge at price s per unit. The retailers sell the product to final consumers at no additional cost to themselves. The inverse demand function faced by retailers is given by:

$$P(Q) = a - bQ \qquad a > 0 \quad b > 0 .$$

Where Q denotes the quantity demanded and P the retailers' price.

Each identical retailer maximises profits by choosing an output level, given the price charged by the monopolist, $s(< a)$, and the output level of other retailers.

(*a*) Write down the profit maximisation problem faced by retailer I.

(*b*) Show that the profit-maximising level of output for retailer I is decreasing in the output of other retailers.

(*c*) Using the assumption that the retailers are identical, solve for the equilibrium level of output for each retailer.

(*d*) Write down the monopolist's profit-maximisation problem and solve for the profit-maximising value of s.