

2015

Paper I

A1. (a)  $(x+1)(x-1)^2$ ; (b)  $\frac{1}{4}\left(\frac{3-x}{(x-1)^2} + \frac{1}{x+1}\right)$ . 2. (a)  $m=e$ ; (b)  $(1, e)$ .

3. (a) circle centre  $(1,0)$  radius 2. 4. (a)  $-\operatorname{cosec}^2 x$ . 5. (a)  $\frac{x^2}{2}\sin x^2 + \frac{1}{2}\cos x^2 + c$ ; (b)

$\frac{\pi}{4} - \frac{1}{2}$ .

6. (a)  $y=1, x=\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$ ; (b)  $-3\cos 3t/\sin t$ . 7.  $1 - \frac{3x}{2} - \frac{9x^2}{8}$ ;  $\frac{-1}{3} < x \leq \frac{1}{3}$ .

8.  $x = \pm \frac{\pi}{2}, \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$ . 9. (a)  $-\sqrt{6}$ ; (b)  $y = \frac{x}{\sqrt{6}} + \frac{1}{2}$ . 10. (a) 500500; (b) 4094.

B11. (a)(i)  $-1, -i$ ; (b)  $\dot{z} = (\dot{r} + i r \dot{\theta}) e^{i\theta}, \ddot{z} = (\ddot{r} - r \dot{\theta}^2 + i(2\dot{r}\dot{\theta} + r\ddot{\theta})) e^{i\theta}$ ; (c)(i) circle centre  $(0,0)$ , radius 2; (ii)  $\dot{r} = \ddot{r} = 0, \dot{z} = 2i\dot{\theta} e^{i\theta}, \ddot{z} = (-2\dot{\theta}^2 + 2i\ddot{\theta}) e^{i\theta}$ ;

(iii) radial =  $\ddot{r} - r\dot{\theta}^2$ , transverse =  $2\dot{r}\dot{\theta} + r\ddot{\theta}$ .

12. (a)(i) exact,  $f = e^{x+y} + c$ ; (ii) not exact; (iii) exact,  $f = x^2 y^3 z^4 + c$ ;

(b)  $g(2m\pi, 0) = 0$  minima,  $g((2m+1)\pi, 0) = 2$  saddles.

13. (b)  $u(x, t) = (t+a)^{\frac{-1}{2}} e^{\frac{-|x+b|^2}{4\lambda|t+a|}}$ ; (c)  $(4\lambda t+1)^{\frac{-1}{2}} \left( e^{\frac{-|x+1|^2}{4\lambda t+1}} + e^{\frac{-|x-1|^2}{4\lambda t+1}} \right), t > \frac{-1}{4\lambda}$ .

14. (a)  $a - a \cos t, a \sin t, \frac{\sin t}{1 - \cos t}$ ; (c)  $a\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ ; (d)  $3\pi a^2$ ; (e)  $8a$ .

15. (b)  $\cosh \sqrt{2} + \frac{(x-2)}{2\sqrt{2}} \sinh \sqrt{2} + \frac{(x-2)^2}{16} \left( \cosh \sqrt{2} - \frac{1}{\sqrt{2}} \sinh \sqrt{2} \right)$ ; (c)(i)  $x - 2x^2 + \frac{17x^3}{6}$ ;

(ii)  $1 + \frac{x^2}{3} - \frac{19x^4}{120}$ .

16. (a)(i)  $3/7, 4/7, 3/7$ ; (ii)  $2/7$ ; (iii)  $1/2, 1/2$ ; (iv)  $6/7$ ; (v)  $12/7$ ; (b)  $\frac{N_B}{N}$ .

17. (a)  $\frac{b^3}{3}$ ; (b)(i)  $\sin^{-1}\left(\frac{x}{2}\right) + c$ ; (ii)  $\ln \sqrt{\left(\frac{x}{2} + 1\right)^2 + 1} - \frac{1}{2} \tan^{-1}\left(\frac{x}{2} + 1\right) + c$ ;

(iii)  $F(k) = \pi^k - k(k-1)F(k-2)$ ;  $F(5) = \pi^5 - 20\pi^3 + 120\pi$ .

18. (d)  $\lambda=1, v=(1 \ 1 \ 1)^T, \lambda=e^{\frac{2i\pi}{3}}, v=\begin{pmatrix} 1 & e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}^T, \lambda=e^{\frac{4i\pi}{3}}, v=\begin{pmatrix} 1 & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} \end{pmatrix}^T$ ;

(e)  $\lambda=1, v=(1 \ 1 \ 1)^T, \lambda=e^{\frac{2i\pi}{3}}, v=\begin{pmatrix} 1 & e^{\frac{2i\pi}{3}} & e^{\frac{4i\pi}{3}} \end{pmatrix}^T, \lambda=e^{\frac{4i\pi}{3}}, v=\begin{pmatrix} 1 & e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix}^T$ ;

(f)  $\frac{1}{2}(A+A^T)$ ; (g)  $1, -\frac{1}{2}, -\frac{1}{2}$ .

19. (a)  $\frac{3}{16} \left[ 4N \left( \frac{-1}{3} \right)^{N+1} - \left( \frac{-1}{3} \right)^N + 1 \right], \frac{3}{16}$ .

20. (c)  $-\alpha r \hat{p} + 2\alpha z k$ ; (e)  $\pm \alpha R A$  (depending on which way round the edge one goes).

Paper II

A1. (a)  $\hat{n} = \frac{1}{\sqrt{30}}(-5 \ 1 \ 2)^T, \frac{2}{\sqrt{30}}$ . 2. (a)  $-2, 1 \pm i\sqrt{3}$ . 3.  $x^2$ , minimum.

4.  $\lambda = -1, v = \frac{1}{\sqrt{2}}(1 \ -1)^T, \lambda = 3, v = \frac{1}{\sqrt{2}}(1 \ 1)^T$ .

5.  $F = (\cos y^5 \sinh z \quad -5y^4 x \sin y^5 \sinh z \quad x \cos y^5 \cosh z)^T$ ,  $\text{curl } F = 0$ . 6. (a)  $2r$ ; (b)  $3a$ .

7.  $y = x^2$ . 8.  $\frac{4}{5} \pi R^5$ . 9.  $\sin x + 2 \sin 2x$ . 10. (a)  $\frac{1}{2^{10}}$ ; (b)  $1 - \frac{11}{2^{10}}$ .

B11. (a)  $\lambda = 1, \mu = -3, p = (3 \ 0 \ 1)^T$ ; (b)(i) the point  $i$ ; (ii) the line  $(t, -1-t, -1-t)$ ; (iii) no points.

12. (b)  $a^2 \pi \sin \frac{\beta}{2}$  from the centre along the line through the centre of mass.

13. (a)  $\mathbf{F}$  is not;  $\mathbf{G}$  is,  $\phi = -x^2 y \cosh z + c$ ; (b) both  $\cosh 1$ ; (c)  $3 \cosh 1 - 3 \sinh 1, \cosh 1$ .

14. (b)  $\frac{2}{T_2}$ ; (c)  $\frac{t}{T_1}$  for  $0 < t < T_1$ ; (d)  $\frac{t^2}{T_2^2}$  for  $0 < t < T_2$ ; (e)  $\frac{t^3}{T^3}$  for  $0 < t < T$ ;

(f)  $R(t) = \frac{t}{T} + \frac{t^2}{T^2} - \frac{t^3}{T^3}$  for  $0 < t < T$ ; (f)  $(\frac{T}{3}, \frac{11}{27})$ .

15. (a)(i) no solutions; (ii)  $f = e^{-2t} + 2e^{-6t} - 3e^{-4t}$ ; (b)  $y = (1 - 2x + 2x^2)e^{-x}, y = 5e^{-2}$ .

16. (a)  $\frac{1}{P} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$  fn of  $y$  only; (i)  $y = kx^{-1/2} - x$ ; (ii)  $y + y^2 \tan x = k$ ;

(b)  $\ln \sqrt{y^2 + 2xy + 2x^2} - 2 \tan^{-1} \left( \frac{y}{x} + 1 \right) = k$ .

17. You might get different versions depending on how you form your "eigenvector equation":

(c)  $\begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \omega^2 m \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ ; (d)  $\lambda = 2k, v = \frac{1}{\sqrt{2}} (1 \ 0 \ -1)^T$ ,

$\lambda = k(2 + \sqrt{2}), v = \frac{1}{2} (1 \ -\sqrt{2} \ 1)^T, \lambda = k(2 - \sqrt{2}), v = \frac{1}{2} (1 \ \sqrt{2} \ 1)^T$ ; (f)  $x_1$  and  $x_3$  are in phase,  $x_2$  is opposite in phase and  $\sqrt{2}$  bigger in magnitude.

18. (a)  $\omega = \frac{2\pi}{T}$ ; (c)  $a_n = 0$  if  $n$  is odd,  $a_n = \frac{2V_0}{\pi(1-n^2)}$  if  $n$  is even,  $b_1 = V_0/2, b_n = 0$  otherwise.

19. (b)  $f = \ln n$ ; (c)  $\sum_{i=1}^n a e^{-b y_i} = 1, \sum_{i=1}^n y_i a e^{-b y_i} = Y$ .

20. (a)  $u_s = \frac{G}{n^2 \pi^2 v} \sin n\pi x$ ; (b)  $\frac{\partial \tilde{u}}{\partial t} = v \frac{\partial^2 \tilde{u}}{\partial x^2}, \tilde{u}(0, t) = 0, \tilde{u}(1, t) = 0, \tilde{u}(x, 0) = \frac{-G}{n^2 \pi^2 v} \sin n\pi x$ ;

(c)  $\tilde{u} = \frac{-G}{n^2 \pi^2 v} e^{-n^2 \pi^2 v t} \sin n\pi x$ ;

(d)  $u(x, t) = \frac{G}{n^2 \pi^2 v} \sin n\pi x (1 - e^{-n^2 \pi^2 v t}), Q(t) = \frac{G(1 - e^{-n^2 \pi^2 v t})(1 - \cos n\pi)}{n^3 \pi^3 v}$ .

- A1. (a)  $(a + b)(a^2 - ab + b^2)$ ; (b)  $x = 2$ . 2. (a)  $-\frac{1}{2}(x - 1)^2 + \frac{1}{2}$ ; (b)  $\frac{\frac{3}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$ .
3. (a)  $10^8 - 4$ ; (b) 0. 4.  $\left[\frac{4\pi}{3}, \frac{5\pi}{3}\right] \pm 2n\pi$ . 5. (a)  $x = \pm \frac{1}{\sqrt{2}}$ . 6. (a)  $y = (2 - x)^{-1}$ .
7. (a)  $-\frac{1}{3} \cot(3x + 1) + c$ ; (b)  $\frac{(n-2)^{n+1}}{n+1} - \frac{(-2)^{n+1}}{n+1}$ . 8.  $-\sin x (\ln 2 + 1)(2e)^{\cos x}$ . 9. -. 10.  $0 \leq k \leq \frac{4}{3}$ .
- B11. (a)  $y = 0$ ; (b)  $z = i$ ; (c)  $b = \frac{(\lambda^2+1)i}{(\lambda^2-1)}$ ,  $c = 1$ , centre is  $\left(0, -\frac{(\lambda^2+1)}{(\lambda^2-1)}\right)$ , radius is  $\frac{2\lambda}{|\lambda^2-1|}$ ;  
 (d)(i)  $|b|^2 = c$ ; (ii)  $|b|^2 > c$ ; (iii) impossible.
12. (b)(i) Min at (0,0), saddles at  $(\pm 1, \pm 1)$ .
13. (a)  $\nabla \cdot \mathbf{v}_1 = 0$ ,  $\nabla \cdot \mathbf{v}_2 = 10y$ ; (b)  $I_1 = \frac{5\pi}{2}bc(a_z b - a_x) + 2bc(a_x - a_y)$ ,  $I_2 = b^3 + 25\pi^2 c^2 b/4$ ;  
 (c)  $\mathbf{v}_1$  is not,  $\mathbf{v}_2$  is conservative; (d)  $\phi = x^2 y + y^3 + z^2 y + c$ .
14. (b)(i)  $\lambda = 1, (0 \ 0 \ 1)^T$ ,  $\lambda = 2, \frac{1}{\sqrt{2}}(1 \ 1 \ 0)^T$ ,  $\lambda = 4, \frac{1}{\sqrt{2}}(1 \ -1 \ 0)^T$ ; (ii)  $\frac{1}{8} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ .
15. (a)(i)  $\frac{1}{2} \ln 2 + \frac{x}{4}(\ln 2 + 1) + \frac{x^2}{8} \left(\ln 2 + \frac{1}{2}\right)$ ; (ii)  $x - \frac{x^3}{3} + \frac{x^5}{5}$ ; (iii)  $\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45}$ ;  
 (b)(i)  $\frac{d^2 g}{df^2} = -\frac{d^2 f}{dg^2} / \left(\frac{df}{dg}\right)^3$ ,  $\frac{d^3 g}{df^3} = 3 \left(\frac{d^2 f}{dg^2}\right)^2 / \left(\frac{df}{dg}\right)^5 - \frac{d^3 f}{dg^3} / \left(\frac{df}{dg}\right)^4$ ;  
 (ii)  $b_1 = 1/a_1$ ,  $b_2 = -a_2/a_1^3$ ,  $b_3 = 2a_2^2/a_1^5 - a_3/a_1^4$ .
16. (a)(i)  $(R - r)/R$ ; (v)  $r^2/2R$ ; (vi)  $r^3/3R - r^4/4R^2$ ; (b)(i)  $(r - t)/R$ ; (ii)  $(R - r)/r$ ; (iii)  $t/R$ .
17. (a)(i)  $y = a_1 x + a_0$ ; (ii)  $\sum_{i=0}^{n-1} a_i x^i$ ; (b)(i)  $A_0 + A_1 e^x + A_2 e^{-x} + A_3 \cos x + A_4 \sin x$ ; (ii)  $-\frac{1}{2} x^2$ ;  
 (iv)  $y = 1 + \frac{\cosh x}{2} - \frac{1}{2} \cos x - \frac{x^2}{2}$ .
18. (a)(iii)  $I = \frac{1}{\alpha}$ ; (b)(ii)  $\frac{(m-2)!(n-2)!}{(m+n-3)!}$ .
19. (a) & (b)  $\begin{pmatrix} \tan \theta \cos \phi & \tan \theta \sin \phi & 1 \\ W \sec^2 \theta \cos \phi & W \sec^2 \theta \sin \phi & 0 \\ -W \tan \theta \sin \phi & W \tan \theta \cos \phi & 0 \end{pmatrix}$ ; (c)  $\frac{1}{W} \begin{pmatrix} 0 & \cos^2 \theta \cos \phi & -\cot \theta \sin \phi \\ 0 & \cos^2 \theta \sin \phi & -\cot \theta \cos \phi \\ W & \cos^2 \theta \tan \theta & 0 \end{pmatrix}$ .
20. (a)(ii) Yes; (iii)  $(y^2 \ z^2 \ x^2)^T$ ; (b)(ii)  $\pi a^3 b/4$ ; (c)  $\pi a^3 b/4$ .

## Paper II

- A1.  $x \geq 1, y = 0$ . 2.  $\mathbf{r}_0 = (1 \ 2 \ -1)^T$ ,  $t = (4 \ 3 \ 2)^T$ . 3.  $\frac{\partial f}{\partial r} = \cos 2\theta \frac{\partial f}{\partial x} + \sin 2\theta \frac{\partial f}{\partial y}$ ,  
 $\frac{\partial f}{\partial \theta} = -2r \sin 2\theta \frac{\partial f}{\partial x} + 2r \cos 2\theta \frac{\partial f}{\partial y}$ . 4(i)  $\left(\frac{35}{36}\right)^2 \frac{1}{36}$ ; (ii)  $\frac{36}{71}$ . 5. (a) Yes; (b) No. 6. (a) Zero; (b)  $\sin \frac{\pi}{8} \cos \frac{\pi}{4}$ .
7. (a)  $\sin x + \sin 2x + \sin 3x$ ; (b)  $\cos x + 2 \cos 2x + 3 \cos 3x$ . 8.  $y = x e^x$ .
9.  $\frac{4}{3} \pi (b^3 - a^3) + 4\pi \left(\frac{1}{c} - \frac{1}{a}\right)$ . 10. (a)  $xy$ ; (b) saddle.
- B11. (a)(i)  $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0$ ; (ii)  $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b}) > 0$ ; (c)  $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b}) > 0$ ,  $\mathbf{r} \cdot (\mathbf{b} \wedge \mathbf{c}) > 0$ ,  $\mathbf{r} \cdot (\mathbf{c} \wedge \mathbf{a}) > 0$ ,  
 $\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a}) < \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$ ; (d)  $\sqrt{6}$ .
12. (a)  $(r, \theta, a(1 - r^2/b^2))$ ; (b)  $\alpha ab^2/4$ ; (d)  $\alpha b^2/2$ ;  
 (e)  $\mathbf{S}_{ABCA} = \left(\left(\frac{2ab}{3}\right) \sin \alpha \quad \left(\frac{2ab}{3}\right) (1 - \cos \alpha) \quad \alpha b^2/2\right)^T$ .
13. (b)(i)  $t = 0$  to infinity; (ii)  $3a/2$ ; (c)  $y^3 + x^3 - 3ayx = 0$ ,  $x = y$ .
14. (a)  $y = \frac{1}{3}(a^2 + x^2) + c(a^2 + x^2)^{-\frac{1}{2}}$ ; (b)(i)  $|x + y - 2| = A e^{-\frac{1}{2}(x+3y)}$ ; (ii) Yes,  $y = 2 - x$ ;  
 (c)(ii)  $x = \frac{B|1-p^2|}{p^2}$ ; (iii)  $y^2 = 2Bx + B^2$ .

15. (b)  $x = \frac{t-s}{t-1}, y = \frac{s+t-2}{t-1}, z = -1$

$t \neq 1$  one solution,  $t = 1, s = 1$  infinitely many solutions,  $t = 1, s = 6$  no solutions.

16. (a)(i)  $0: \frac{1}{10}, 1: \frac{3}{5}, 2: \frac{3}{10}$ ; (ii)  $\mu = \frac{6}{5}, \sigma = \frac{3}{5}$ ; (iii)  $\frac{3}{5}$ ; (iv)  $\frac{7}{10}$ ; (b)(i)  $\frac{13}{30}$ ; (ii)  $\frac{12}{13}$ .

17. (b)(i)  $\mathbf{v} = (2y, -x)$ ; (v)  $n = m = 2, A = \frac{1}{2}, f(x^2 + 2y^2) + \frac{1}{2}x^2y^2$ ;

(vi)  $\frac{1}{2}(x^2 + 2y^2 - 2) + \frac{1}{2}x^2y^2$ .

18. (b)(i) Fourier coeffs are sin:  $b_n \cos \frac{2\pi nl}{L}$  and cos:  $-b_n \sin \frac{2\pi nl}{L}$ ; (c)(i)  $-\frac{K}{n\pi}(-1)^n$ ;

(ii)  $\sum_{n=1}^{\infty} \frac{K(-1)^n}{n\pi} \sin \frac{2\pi nvt}{L} \cos \frac{2\pi nx}{L} - \frac{K(-1)^n}{n\pi} \cos \frac{2\pi nvt}{L} \sin \frac{2\pi nx}{L}$ .

19. (a) a cube; (c)(i)  $\frac{1}{2}$ ; (ii)  $\cos a$ ; (iii)  $e^{2a}$ .

20. (b)  $f = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda ct + D \sin \lambda ct), f = (Ax + B)(Ct + D),$

$f = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{\lambda ct} + De^{-\lambda ct})$ ; (c)  $f = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( C_n \cos \frac{n\pi ct}{L} + D_n \sin \frac{n\pi ct}{L} \right)$ ;

(d)  $f = \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L}$ .

A1. (a)  $-\ln 2$ ; (b)  $y=2e^{-2}$ . 2. (a)  $-\frac{2}{3}\cos^3 x+c$ ; (b)  $\frac{x^2}{2}\ln x - \frac{x^2}{4}+c$ .

3. (a)  $2^x \ln 2$ ; (b)  $-\tan^2 t$ . 4. (a)  $-\frac{2}{3}$ ; (b)  $y=-\frac{2}{3}x+\frac{5}{3}$ .

5. (a)  $x=2$ ; (b)  $(x-2)(2x+1)(x+3)$ . 6. (a)  $(1+2x)^4$ .

7. (a)  $\sin^2 \frac{\pi}{12} = \frac{1}{2} \left( 1 - \sqrt{1 - \sin^2 \frac{\pi}{6}} \right)$ ; (b) 0.067. 8. (a)  $x=\frac{\pi}{2}$  or  $x=\frac{\pi}{6}$ ; (b)  $x=\frac{3\pi}{2}$ .

9. (a)  $y=\frac{1}{c-x}$ ; (b)  $y=\frac{1}{2-x}$ . 10. (a)  $OA=5$ ,  $OB=13$ ; (b) 63/65.

B11. (a)  $z=re^{i\theta}$ ; (b) (i)  $\Re = \frac{1}{2} \ln(x^2+y^2)$ ,  $\Im = \tan^{-1} \frac{y}{x} + 2n\pi$ ; (ii)  $\Re = \frac{x}{x^2+y^2}$ ,  $\Im = \frac{-y}{x^2+y^2}$ ;

(iii)  $\Re = 0$ ,  $\Im = 2y(x^2+y^2)$ ; (iv)  $\Re = \cosh x \cos y$ ,  $\Im = \sinh x \sin y$ ;

(c) (i) circles centred on origin; (ii) circles just touching the origin, centres on x-axis;

(d)  $x=\cosh^{-1} 2$ ,  $y=(2n+1)\pi$ ,  $n$  integer.

12. (a)(ii)  $\left( \frac{\partial x}{\partial y} \right)_z = \frac{xz}{\cosh(x+z) - yz}$ ,  $\left( \frac{\partial x}{\partial z} \right)_y = \frac{xy - \cosh(x+z)}{\cosh(x+z) - yz}$ ,  $\left( \frac{\partial y}{\partial z} \right)_x = \frac{\cosh(x+z) - xy}{xz}$ ;

(b) All  $a$ ,  $b$  not both zero; (ii)  $f = \tan^{-1} \left( \frac{by}{ax} \right) + c = -\tan^{-1} \left( \frac{ax}{by} \right) + k$ ; (iii)  $y=Kx$ .

13. (a)  $y=ke^{-2x}+x^2-x+c$ ; (b)  $y=e^{-\int P(x)dx} \int Q(x)e^{\int P(x)dx} dx$ ;  $y=e^{-3x^2/2} (3(x^2+1)^{3/2} - 2)$ .

14. (b)(i) Minimum at  $(0,0)$ , saddles at  $\left( \frac{\pm 1}{\sqrt{-a}}, 0 \right)$  if  $a < 0$ ; (iii)  $y = \frac{\pm 1}{\sqrt{2}} (x^2 - 1)$ .

15. (b)  $9^{1/3} \approx 2 + \frac{1}{12} - \frac{1}{(18)(16)}$ ,  $\max R_n = \frac{5}{81(256)}$ ; (c)(i)  $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4$ ;

(ii)  $1 - \frac{x^2}{2} - x^3 + \frac{5x^4}{24}$

16. (a)  $N=(13)(48)$ ; (b) Binomial  $\binom{100}{n} p^n (1-p)^{100-n}$ ,  $\sum_{n=51}^{100} \binom{100}{n} p^n (1-p)^{100-n}$ ;

(c) (ii)  $\frac{1}{2} \tanh \left( \frac{\mu}{2s} \right)$ .

17. (a) (i)  $\ln(\ln x) + c$ ; (ii)  $\cosh x + \operatorname{sech} x + c$ ; (b)  $I_2 = \frac{3\sqrt{\pi}}{4}$ ,  $I_3 = \frac{15\sqrt{\pi}}{8}$ .

18. (a)  $\mathbf{A}\mathbf{A}^T = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ ,  $\mathbf{A}^T\mathbf{A} = \frac{1}{3} \begin{pmatrix} 4 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & 4 \end{pmatrix}$ ; (b)  $\lambda=1$   $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,

$\lambda=2$   $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ; (c)  $\sigma=0$   $\mathbf{v} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ ,  $\sigma=1$   $\mathbf{v} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,

$\sigma=2$   $\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ; (d)  $\mathbf{0}$ ,  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

19. (a); (d) (i) diverges (ii) converges (using appropriate integral approximations is the least horrid method).

20. (a)  $y^3=x^3+cx$ ; (b)  $x^2+2y^2=k$ .

Paper II

- A1. (a)  $90^\circ$ ; (b)  $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}^T$ . 2. (a)  $\Re = e^{-y}(x \cos x - y \sin x)$ ,  $\Im = e^{-y}(x \sin x + y \cos x)$ .
3. (a)  $\lambda = \pm a$ ; (b)  $\lambda = a$ :  $\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ,  $\lambda = -a$ :  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . 4.  $x^3 - x^5$ . 5.  $\frac{3}{8} + \frac{1}{2} \cos 2x + \dots$
6.  $\begin{pmatrix} \sin(y-z) & \sin(z-x) & \sin(x-y) \end{pmatrix}^T$ . 7. (a)  $y = A \cos 3x + B \sin 3x$ ; (b)  $y = \cos 4x$ .
8. (a)  $\mathbf{0}$ ; (b)  $\frac{1}{12}$ . 9. (a)  $u''y^{-1} - \frac{1}{2}u'y^{-\frac{3}{2}}$ . 10. (a)  $\frac{p+q}{2}$ ; (b)  $\frac{p}{p+q}$ .
- B11. (b)  $|\mathbf{a}-\mathbf{b}| < p+q$  and  $|\mathbf{a}-\mathbf{b}| > |q-p|$ .
12. (a)(ii)  $4\pi\rho_0 h_0(R^2 + 2Rh_0 + 2h_0^2)$ ; (b)(ii)  $A=1, B=\frac{1}{3}$ .
13. (d)  $\phi = x^2y - y^2 - xz^3 - z + c$ ; (e)  $-126$ .
14. (a)  $\mu = \sum_{x=0}^{\infty} xP(x)$ ,  $\sigma^2 = \sum_{x=0}^{\infty} x^2P(x) - \mu^2$ ; (b)(ii)  $\sum_{x=0}^K \frac{e^{-\lambda}\lambda^x}{x!}$ ; (c) (i)  $(1-e^{-\lambda})^{10}$ ; (iii)  $50\lambda^2 e^{-10\lambda}$ .
15. (a)(i)  $\mu = y^{-3}$ ,  $x^2 - xy + ky^2 = 0$ ; (ii)  $\mu = \cos x$ ,  $y \cos^2 x + y^4 \cos^3 x = c$ ;  
 (b)  $y = x \pm 2x^{1/2}$  if  $x > 0$ , or  $y = Ax + 1/(A-1)$ .
16. (b)(i)  $-\frac{44}{3}$ ; (iii)  $-\frac{44}{3}$ .
17. (a)  $\frac{1}{2} \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}$   $\lambda=1$   $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\lambda=2$   $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ,  $\lambda=4$   $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ; (b)  $\mathbf{y} = K \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T e^{-t}$ ,  
 $\mathbf{y} = K \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T e^{-2t}$ ,  $\mathbf{y} = K \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T e^{-4t}$ ; (c)  $\mathbf{y}_0 = K^{-1} \mathbf{f}_0$ ,  $\mathbf{y}_0 = \begin{pmatrix} 1 & 1 & 1/4 \end{pmatrix}^T$ ;  
 (d)  $\mathbf{y} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T e^{-t} + \frac{1}{4} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T e^{-4t}$ .
18. (a)  $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$ ; (b)  $p=3$ .
19. (b)  $V_p = \frac{R^2 H}{2}$ ; (c) the cylinder.
20. (a)(i)  $y = \left(1 - \frac{1}{2} \ln(1+x)\right)^2$ ; (ii)  $y = (1-3x)^{-\frac{1}{3}} e^{-\frac{x}{3}}$ ; (b)  $T = C e^{-\left(1 + \frac{n^2 \pi^2}{L^2}\right)t}$ ;  $X = B \sin\left(\frac{n\pi x}{L}\right)$  so  
 general solution is  $u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(1 + \frac{n^2 \pi^2}{L^2}\right)t} \sin\left(\frac{n\pi x}{L}\right)$ .

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Paper I

- A1. (a)  $x=y/\log_{10} 2$ ; (b) 79 . 2. (a)  $x=1, -1/2, -5$ ; (b)  $-3/2-\sqrt{5}/2 < x < -3/2+\sqrt{5}/2$  .
3.  $(t, \theta)=(0, \pi/2), (-1, 0)$ ; . 4. (a)  $1/9$ ; (b)  $e^{\sin^2 x} + c$ ; 5. (a)  $\theta=2\pi/3$ ; (b)  $3\sqrt{3}/4\pi$  (Area B).
6. (a)  $\sqrt{3}$ ; (b)  $\cos\theta=1/3$  . 7. (a) 1.094; (b)  $9/2$  . 8.  $x=\pi/4, y=e^{\pi/4}/\sqrt{2}$  .
9. centre  $(-1/2, -1)$  radius  $3/2$  . 10. (a)  $(1, 4)$ ; (b)  $y=-2$  .
- B11. (a)  $|z|=1/\sqrt{2}$ ,  $\arg(z) = 5\pi/12$ ,  $|w|=\sqrt{1/4 \ln^2 2 + \pi^2(5/12+2n)^2}$ ,  
 $\arg(w) = -\tan^{-1}((5/12+2n)\pi/\frac{1}{2} \ln 2) + \pi$ ;
- (b)  $z=-i\pi/4+in\pi$ ; (c)  $16\cos^5\theta-20\cos^3\theta+5\cos\theta$  .
12. (a)(ii)  $(\sqrt{2}-1)\sqrt{\pi}/2$ ; (iii)  $(\sqrt{2}-1)\sqrt{\pi}$ ; (b)(i)  $4\pi abc/3$  (ii) zero.
13. (a)  $y=(\ln|x|+c)/x$ ; (b)  $y=\frac{1}{4}e^{3x}+ce^{-x}$ ; (c)  $5x^2/2+4xy-2y^4=c$ ; (d)  $\ln|y|=2(x/y)^{1/2}+c$  or  $y=0$  .
14. (a)  $y=1/4x$ ; (c)  $(0,0), \pm(\sqrt{3}/2, \sqrt{3}/2), \pm(1/2, -1/2)$ ; (d)  $(0,0)$  saddle,  $\pm(\sqrt{3}/2, \sqrt{3}/2)$  minima,  $\pm(1/2, -1/2)$  maxima.
15. (b)  $1/a_0 - a_1 x/a_0^2 + (a_1^2/a_0^3 - a_2/a_0^2)x^2$ ; (c)  $1+x^4/6$ ; (d)  $\ln 2 + 1/2x + 3/(8x^2)$
16. (a)  $y_i = x_0 \tan \theta_i$ ; (b)  $g(\theta) = 1/\pi$  for  $-\pi/2 \leq \theta \leq \pi/2$ , zero otherwise,  $G(\theta) = \theta/\pi + 1/2$  for  $-\pi/2 \leq \theta \leq \pi/2$ ; (c)  $F(y) = (1/\pi) \tan^{-1}(y/x_0) + 1/2$ ; (f)  $x_0 \ln 3/\pi$ ; (g)  $x_0 \ln 3/\pi$  .
17. (a)  $(1/2) \ln |1 - \cot^2(x)| + c$ ; (c)  $E_{n+1} = (2n-1)E_n/2n$ ,  $E_4 = 5\pi/32$  .
18. (a)  $\text{Det } \mathbf{A} = 0$ ,  $\text{Tr } \mathbf{A} = 6$ ,  $\text{Det } \mathbf{A}^T = 0$ ,  $\text{Det } \mathbf{A}^2 = 0$ ; (b)  $\lambda = -6$   $\mathbf{u} = (1/\sqrt{2})(1 \ 0 \ -1)^T$ ,  
 $\lambda = 0$   $\mathbf{u} = (1/\sqrt{6})(1 \ 2 \ 1)^T$ ,  $\lambda = 12$   $\mathbf{u} = (1/\sqrt{3})(1 \ -1 \ 1)^T$ ; (c)  $\mathbf{u} = k(1 \ 2 \ 1)^T$   $k \in \mathbb{R}$  .
19. (a)(ii)  $m=0$  not continuous or differentiable,  $m=1$  continuous but not differentiable,  $m=2$  continuous and differentiable; (b)(i) diverges; (ii) converges.
20. (a)  $\tan^{-1}\alpha/\alpha$  .

Paper II

A1. (a)  $(1/\sqrt{5})(0 \ 1 \ 2)^T$ ; (b)  $2/\sqrt{5}$ . 2.  $z=in\pi$ ,  $z=i\pi(2n+1)$ .

3.  $2 \cosh \phi$  or  $2 \sinh \phi$ . 4.  $x^2-3x^4/2$ . 5.  $\ln x-t$ . 6. (a)  $\mathbf{r} = \frac{\cos f(r)}{r} \frac{df}{dr}$ ; (b)  $\frac{z}{r} \frac{df}{dr}$ .

7. (a)  $y=xe^{-x}$ ; (b)  $y=3/(x^3+3)$ . 8. (a) 0; (b)  $\pi^2$ . 9. (a)  $(\pm\sqrt{\frac{3}{2}}, 0)$ . 10. (a)  $\alpha=1/2$ ; (b)  $\pi/2$ .

B11. (b)  $l=\sqrt{15}$ ; (c)  $\mathbf{x}=(1/5)(2\mathbf{b}-3\hat{\mathbf{n}}-\hat{\mathbf{n}}\times\mathbf{b})$ .

12. (a)  $\sin \phi$ ; (b)  $(\frac{\partial \phi}{\partial y})_x = \frac{\cos \phi}{r}$ ; (c)  $f_y = g_r \sin \phi + g_\phi \frac{\cos \phi}{r}$ ;

(d)  $\frac{\partial^2 f}{\partial y^2} = s^2 g_{rr} + \frac{2sc}{r} g_{r\phi} - \frac{2sc}{r^2} g_\phi + \frac{c^2}{r} g_r + \frac{c^2}{r^2} g_{\phi\phi}$ ,  $\nabla^2 f = g_{rr} + \frac{g_r}{r} + \frac{g_{\phi\phi}}{r^2}$ ;

(e)(i)  $f=2xy$ ,  $\nabla f=(2y, 2x)$ ; (iii)  $3\sqrt{2}$ .

13. (a)(i)  $\int \mathbf{F} \cdot d\mathbf{r}=2$ ,  $\int \mathbf{G} \cdot d\mathbf{r}=0$ ; (ii)  $\int \mathbf{F} \cdot d\mathbf{r}=2\pi$ ,  $\int \mathbf{G} \cdot d\mathbf{r}=0$ ,  $\mathbf{G}$  is conservative,  $\Phi=x^2y^2+c$ ; (b)(i) start  $(1,0,0)$  end  $(0,1,2\pi/3)$ , helix; (ii)  $5/6$ .

14. (a)(i)  $G \cap x_0, G \cap x_1, G \cap x_2, D \cap x_0, D \cap x_1, D \cap x_2$ ; (ii)  $pg_0$ ; (iii)  $pg_0+(1-p)d_0$ ;

(iv)  $d_1(1-p)/(pg_1+(1-p)d_1)$ ; (v)  $(g_1+g_2)p/((g_1+g_2)p+(d_1+d_2)(1-p))$ ;

(b)(i)  $p \sum_{i=k+1}^n g_i + (1-p) \sum_{i=k+1}^n d_i$  if  $k < n$ ; (ii)  $p \sum_{i=0}^k g_i / [p \sum_{i=0}^k g_i + (1-p) \sum_{i=0}^k d_i]$ ;

(iii)  $p=1/(1+g_0)$ ,  $\min p=1/2$  when  $g_0=1$ .

15. (a)(i)  $y=e^x(A \cos x + B \sin x) + (x+1)^2$ ; (ii)  $y=(A+x)e^{-x} + Be^{-2x}$ ; (b)  $y=Ax \ln x + Bx$ .

16. (a)(i)  $6(\mathbf{a} \cdot \mathbf{b})$ ; (ii)  $\mathbf{0}$ ; (iii)  $-2(\mathbf{a} \cdot \mathbf{b})$ ; (iv)  $\mathbf{a} \times \mathbf{b}$ ; (b)(i)  $|\mathbf{a} \times \mathbf{b}|^2/2$ ; (ii)  $2\pi R^3(\mathbf{a} \cdot \mathbf{b})$ .

17. (a)  $\begin{pmatrix} 1 & \mu & 0 \\ 1 & -1 & 3 \\ 2 & -2 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ ; (b)  $c_i = i^{\text{th}}$  column of  $\mathbf{A}$ ; (c)  $\mu=-1, 6$   $\mu=-1: \lambda(1 \ 1 \ 0)^T$ ,

$\mu=6: \lambda(-6 \ 1 \ 7/3)^T$ ; (d)  $\lambda=7$ , (eg)  $(1 \ 0 \ 2)^T + k(1 \ 1 \ 0)^T$ .

18. (b)  $a_n$  and  $b_n$  are Fourier coeffs of  $f$ ; (c)  $a_n=0$ ,  $b_n=4/(n\pi)$  if  $n$  is odd,  $b_n=0$  otherwise; (c)

$y_{f_0} = \sum_{n \text{ odd}} r^4 \frac{4}{n\pi} \sin n\theta$ ; (d)  $p=2$ .

19. (a)(i)  $x=2/(4-a^2)$ ,  $y=a/(4-a^2)$ ,  $z=(4-3a^2)/(4-a^2)^2$ , no solutions if  $a=\pm 2$ ;

(ii) Hessian is  $\begin{pmatrix} -2 & a \\ a & -2 \end{pmatrix}$ , so point is a max if  $4-a^2 > 0$  and a saddle if  $4-a^2 < 0$ .

20. (a)(i)  $u=\xi\eta/4+A(\xi)+B(\eta)$ ; (ii)  $a=b=d=0, c=-1/2$ ; (b)  $\frac{\partial^2 z}{\partial x^2} = \frac{2xz}{(z^2+x)^3}$ ,  $\frac{\partial^2 z}{\partial y^2} = \frac{-2z}{(z^2+x)^3}$ .



## Paper I

- A1.  $-2 \leq x \leq 3/2$ . 2. (2,1). 3. (a)  $c = \sqrt{3/2} - \pi/6$ ; (b)  $-1$ . 4. (a)  $\sqrt{2}/2$ ; (b)  $1/\sqrt{5}$ .  
 5.  $\pi/6$  or  $5\pi/6$ . 6.  $I = y^3 \ln y/3 - y^3/9 + 1/9$ . 7. (a) 10300; (b) 1364. 8.  $a=2, b=-1$ .  
 9. (a) 0; (b)  $8|\mathbf{a}||\mathbf{b}| = \pi(|\mathbf{a}|^2 + |\mathbf{b}|^2)$ . 10. (a) 2; (b)  $(\sqrt{3}-5, 1-5\sqrt{3})$ .  
 B11. (a)  $8(\cos^6 \theta - \sin^6 \theta)$ ; (b)  $\Re = \frac{\tan x(1 - \tanh^2 y)}{1 + \tan^2 x \tanh^2 y}$ ,  $\Im = -\frac{\tanh y(1 + \tan^2 x)}{1 + \tan^2 x \tanh^2 y}$ ;  
 (c)  $(\frac{x}{2})^2 + (\frac{y-1}{2/3})^2 = 1$  ellipse; (d)  $r = e^\theta$ .  
 12. (a)  $R = \left(\frac{Ve}{2\pi(e-2)}\right)^{1/3}$ ; (b)  $\frac{56}{15}bHL$  (c)  $\pi b D^2 \left(1 - \frac{D}{4H}\right)$ .  
 13. (a)  $x^2 + xe^y - \sin y = e^{\pi/2}$ ; (b)(i)  $y = ce^{-x^2/2} - 5e^{-x^2}$ ; (ii)  $y = (x+1)x^{-x}$ .  
 14. (a)(ii)  $dm = -.04 m_A$ , A has larger rest mass (b)  $x^2 \frac{\partial^2 u}{\partial x^2} = x^2 y^2 \phi'' + \sqrt{xy} \left(\frac{-1}{4} \psi + \frac{y}{x} \psi' + \frac{y^2}{x^2} \psi''\right)$ .  
 15. (b)(i)  $1 - \frac{\pi^2 x^2}{8}$ ; (ii)  $(x+1) - (x+1)^2 + \frac{7}{6}(x+1)^3 - \frac{7}{6}(x+1)^4$ ; (iii)  $x^3 - \frac{x^5}{2} - \frac{x^6}{2}$ .  
 16. (a)  $4\alpha, 6\alpha, \beta, 3\beta, 6\alpha, 9\alpha, 3\beta, 3\beta$  where  $\alpha = r/25, \beta = (1-r)/10$ ; (b)(i)  $2/5$ ; (ii)  $2/5$ ;  
 (iii)  $2/5$ ; (c)(i)  $(5+3r)/50$ ; (ii)  $8r/(5+3r)$ ; (d)  $(N-1)r/(N-r)$ .  
 17. (a)(i)  $\frac{1}{8}e^{4x} + \frac{1}{4}e^{-2x} + c$ ; (ii)  $\frac{-\arctan x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + c$ ; (b)  $\ln 6$ ;  
 (c)  $I_n = -2n\pi^{2n-1} - 2n(2n-1)I_{n-1}$ ,  $I_3 = -6\pi^5 + 120\pi^3 - 720\pi$ .  
 18. (a)(i)  $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$ ; (ii) For  $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $\mathbf{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ , for  $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ ;  
 (b)(i)  $\mathbf{M} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$ ; (ii)  $\alpha = \frac{\bar{y}\bar{x}^2 - \bar{x}\bar{y}\bar{x}}{\bar{x}^2 - \bar{x}^2}$ ,  $\beta = \frac{\bar{x}\bar{y} - \bar{x}\bar{y}}{\bar{x}^2 - \bar{x}^2}$ .  
 19. (a)(i) diverges; (ii) converges to 4.  
 20. (b)  $\int_{\sin x}^{\cos x} -t^4 e^{-xt^4} dt - \sin x e^{-x \cos^4 x} - \cos x e^{-x \sin^4 x}$ ; (c)  $-\frac{\pi}{4} \frac{1}{\sqrt{pq}} \left(\frac{1}{p} + \frac{1}{q}\right)$ .

## Paper II

- A1. (a)  $\frac{1/2(1+i)}{z+i} + \frac{1/2(1-i)}{z-i}$ . 2. (a)  $x^{3n}/n!$ ; (b)  $x^{3n+1}/(3n+1)n!$ . 3.  $y = 3t/5 + ct^{-4}$ .  
 4. (a)  $I_{n+1} = x(\ln x)^{n+1} - (n+1)I_n$ ; (b)  $x \ln x - x + c$ . 5. (a) (1, -1); (b) saddle point.  
 6. (a)  $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$ ; (b) 1.  
 7.  $3/2$  or  $1/2$ . 8.  $\frac{1}{4} \sin 2x + \frac{1}{8} \sin 4x$ . 9.  $\frac{\pi}{3}(4a^3 + 3a^4)$ . 10. (a)  $(0, 0, \pi a^2/2)$ ; (b)  $\frac{1}{\sqrt{3}} \frac{\pi a^2}{2}$ .  
 B11. (a)(ii)  $1/4$ ; (iii)  $\mathbf{f} + \mathbf{g} - \mathbf{e}$ ; (b)  $\hat{\mathbf{n}} = (1, 1, 1)/\sqrt{3}$ ,  $x + y + z = 1$ ,  $1/\sqrt{3}$ ; (c)  $\sqrt{3}/8$ .  
 12. (a)  $(0, 0), (0, 1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)$ ; (b) min, saddle, saddle, saddle respectively.  
 13. (a)  $\alpha^2 + 2\alpha + \frac{5}{6}\beta$ ; (b)  $\alpha^2 + 2\alpha + \frac{29}{35}\beta$ ; (c)  $\beta = 0$ ; (d)  $\phi = \alpha x^2 + \alpha^2 xy + \alpha xz$ ,  $\alpha = -1 \pm \sqrt{2}$ .

14. (a)  $\int_a^b f(x) dx$ ; (b)(i)  $\int_{a-c}^{b-c} f(x) dx$ ; (ii)  $t(u)=f(u-c)$ ; (c)(i)  $f(z-y)dz$ ;

(d)(i)  $g(y)=1$  for  $-1/2 \leq y \leq 1/2$ , zero otherwise; (ii)  $h(z)=\frac{1}{\pi}(\arctan(z+1/2)-\arctan(z-1/2))$ ;  
(iii) both zero.

15. (a)(ii)  $A=\pi, B=e^\pi$ ; (ii)  $y=(A+Bx+x^3/3)e^{-x}$ ; (b)(i)  $\frac{d^2u}{dt^2}+2\frac{du}{dt}+2u=0$ ;

(ii)  $u(t)=e^{-t}(A \cos t+B \sin t)$ ; (iii)  $e^{-t}(\cos t-\sin t)$ .

16. (a)  $\pi(a^2b+ab^2+a^3)$ ; (b)  $e^{-1}$ .

17. (a)  $\lambda=-1: \mathbf{x}=(1 \ -1 \ 0)^T$ ,  $\lambda=1: \mathbf{x}=(0 \ 0 \ 1)^T$ ,  $\lambda=3: \mathbf{x}=(1 \ 1 \ 0)^T$ ;

(b)(i)  $|\mathbf{A}||\mathbf{B}|$ ; (ii)  $\alpha^n/|\mathbf{A}|$ ; (iii)  $\left(\prod_i \beta_i\right)\left(\prod_j \gamma_j\right)|\mathbf{A}|$ ; (c)(i)  $\mathbf{Q}=\mathbf{E}\mathbf{D}\mathbf{E}^{-1}$ ; (ii)  $\mathbf{V}_k=\mathbf{V}_0\mathbf{E}\mathbf{D}^k\mathbf{E}^{-1}$ .

18. (a)  $a_n=\frac{2(-1)^n \sinh 1}{1+n^2\pi^2}$ ,  $b_n=0$ ;  $p=2, q=\pi$ ; (b)  $2 \sinh(1) \left( \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n \pi \sin(n \pi x)}{1+n^2\pi^2} \right)$ .

19. (a)  $h=(4V/\pi)^{1/3}$ ,  $r=(V/2\pi)^{1/3}$ ,  $S=6\pi(V/2\pi)^{2/3}$ ; (b)  $h=4R/3$ ,  $r=2\sqrt{2}R/3$ ,

$V_{\text{cone}}=\frac{8}{27}V_{\text{sphere}}$ ; (c)  $x_i=a^{1/n}$ , AM  $\geq$  GM.

20. (a)(i)  $Ae^{\lambda(x^2+2y)}$ ; (ii)  $Ax^{2\lambda}y^\lambda$ ; (b)  $F'=-2HF\kappa$ ,  $H'=-4H^2\kappa$ ,  $T(x,t)=\frac{T_0L}{\sqrt{4\kappa t+L^2}}e^{-\left(\frac{x^2}{4\kappa t+L^2}\right)}$