

A1.  $x^2+x+1$  . 2.  $5\pi/4 \leq \theta \leq 3\pi/2$  . 3.  $\sin(1/x) - (1/x)\cos(1/x)$  . 4.  $0 < a < 1$  .

5. (a)  $t^2+2t-1=0$  ; (b)  $\sqrt{2}-1$  . 6.  $-(x/3)(3-2x)^{3/2} - (1/15)(3-2x)^{5/2} + c$  . 7.  $2\pi/3$  .

8. -. 9.  $ar(1-r^{100})/(1-r)$ ,  $r \neq 1$  . 10. 1.42.

B11. (a)(i)  $\operatorname{Re} = 2/5$ ,  $\operatorname{Im} = 1/5$ ; (ii)  $\operatorname{Re} = \cos(1)$ ,  $\operatorname{Im} = \sin(1)$ ; (iii)  $\operatorname{Re} = 0$ ,  $\operatorname{Im} = \pi/2 + 2n\pi$  ;

(iv)  $\operatorname{Re} = 0$ ,  $\operatorname{Im} = 1$ ; (b)  $\left(\frac{y-1}{a/2}\right)^2 + \left(\frac{x}{\sqrt{(a^2-4)}/2}\right)^2 = 1$  ellipse; (i)  $\left(\frac{y-1}{2}\right)^2 + \left(\frac{x}{\sqrt{3}}\right)^2 = 1$  ;

(ii)  $x=0, 0 \leq y \leq 2$  .

12. (a)(i)  $\sqrt{\pi}$  ; (ii)  $3a^2\pi/2$  (b)(i)  $\pi\sqrt{\pi}$  (ii)  $\frac{2\pi}{3}(1-\cos\alpha)$  .

13. (a)(i)  $\sin x$  ; (ii)  $y = \frac{-\cos^3 x}{3\sin x} + \frac{c}{\sin x}$  ; (b)(i)  $y_c = e^{-\sqrt{3}x/2}(A \cos 3x/2 + B \sin 3x/2)$  ;

(ii)  $y_{p,1} = (1/3)e^{-\sqrt{3}x}$  ; (ii)  $y_{p,2} = x/3 - 1/3\sqrt{3}$  ;

(iv)  $y = e^{-\sqrt{3}x/2} \left( (2/3\sqrt{3} - 1/3)\cos(3x/2) + (2\pi/3)e^{\sqrt{3}\pi/2}\sin(3x/2) \right) + e^{-\sqrt{3}x}/3 + 2x/3 - 2/3\sqrt{3}$  .

14. (a)  $f = 1 + (7/2)\ln 3$  at  $(0,0)$ ,  $f = (7/2)\ln 3$  at  $(-1,0)$ ,  $f = 1/4 + (7/2)\ln(7/2)$  at  $(0, \pm 1/\sqrt{2})$  ;

(b) All stationary points are minima except for  $(0,0)$  which seems to be a maximum.

15. (b)(i)  $2 + 1/80 - 1/(50(128))$  ; (c)(i)  $1 - x^2/3 + x^4/18$  ;

(ii)  $(x-1) - (1/2)(x-1)^2 + (1/2)(x-1)^3 - (1/2)(x-1)^4 + \dots$  .

16. (a)(i) Both  $1/2$ ; (ii)  $4/9$ ; (iii)  $4/9$ ; (iv)  $13/18$ ; (v)  $3.5$ ; (b)(i)  $\binom{10}{n} p^n (1-p)^{10-n}$  ; (ii)  $\sqrt{10p(1-p)}$  .

17. (a)(i)  $\ln(2 - \cos x) - \ln(3 - \cos x) + c$  ; (ii)  $-\ln x/(3x^3) - 1/(9x^3) + c$  ;

(iii)  $(x/2)\sqrt{1-x^2} + (\arcsin x)/2 + c$  ; (b)  $J = c^4/4$  .

18. (a)(i)  $(0, \pm 1, 0), (0, \pm 1, 2), (\pm 1, \pm 1, 1)$  ; (ii)  $(0, -1, 0), (0, -1, 2), (\pm 1, 1, 1)$  ; (b)  $\lambda = \pm 3$  ;

(c)(i)  $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  or  $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ,  $C = \begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix}$  ; (ii)  $\lambda = 5, \mathbf{v} = (1/\sqrt{10}) \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ,

$\lambda = -5, \mathbf{v} = (1/\sqrt{10}) \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  .

19. (a) 3; (b)  $(b(\cos \theta)/a, b(\sin \theta)/a, 1) r dr d\theta$  ; (c)  $\pi b a^2$  ; (d)  $4\pi a^3$  ; (e)  $\pi b a^2$  .

20. (a) 1, 2/3; (c)  $\phi = xy + yz + zx + c$  ; (e) three straight line segments, parallel to  $x, y, z$  axes respectively, works.

## Paper I

A1.  $6x^2 \sin x^3 \cos x^3 - 4 \sin x^3 \sin 2x + 6x^2 \cos x^3 \cos 2x - 4 \cos 2x \sin 2x$ .

2.  $(x-2)(x+1)$ ,  $-1 < x < 2$ . 3.  $\theta = \pi/4, 3\pi/4$ . 4. 4, 4/3. 5. Max  $(1/2)e^{-1/2}$ , Min 0.

6.  $-\sqrt{2}/\sqrt{3}, 1/\sqrt{3}$ . 7. Three. 8.  $-1/4 e^{-x^4} + c$ . 9.  $2t/(1-t^2), (1+t^2)^2/(1-t^2)^2$ .

10.  $y = -9x/16 + 25/16$ .

B11. (a)(i) Re=-15/4, Im=2; (ii) Re=161/16, Im=-15;

(b)(i) hyperbolae  $z: (1/2, 2)$  to  $(2, 1/2)$ ,  $z^*: (1/2, -2)$  to  $(2, -1/2)$ ; (ii) horizontal line  $(-15/4, 2)$  to  $(0, 2)$  to  $(15/4, 2)$ ; (iii) parabola  $(161/16, -15)$  to  $(0, -8)$  to  $(-4, 0)$  to  $(0, 8)$  to  $(161/16, 15)$ ; (c)  $\text{Re} + 4 = (\text{Im}/4)^2$ .

12. (a)(i)  $f(x) = 1 - x(1 - e^{-1/x})$ ; (ii)  $R/2r_0$  and  $1 - r_0/R$ ; (b)  $\pi a^2 d/2$ .

13. (a)(i)  $y = A \cos x + B \sin x$ ; (ii)  $y = 1/2 + 1/6 \cos 2x$ ;

(iii)  $y = 1/2 + 1/6 \cos 2x - 2/3 \cos x - 1/3 \sin x$ ; (b)  $y = 0$  or  $y = (ce^{-8x} - 2)^{-1/4}$ .

14. (a)(i)  $dF = Gm_2 dm_1/r^2 + Gm_1 dm_2/r^2 - 2Gm_1 m_2 dr/r^3$ ; (ii) 1%; (b)  $-2\sqrt{2}c/9$ ; (c)  $r$ .

15. (c)  $5/4 + 3/8(x-2) + (x-2)^2/16 - (x-2)^3/32$ ; (d)  $x < \ln 2$ ? (am not sure about this).

16. (a) Mean = 10.5, Var = 35/4; (b)(i) 1/36; (ii) 5/9; (iii) 5/12; (iv) 5/108; (c) 1/10.

17. (a)  $f(x)$ ; (b)(i)  $F = \ln(-x)$ ; (ii)  $G = \ln|x| + c$ ; (c)(i)  $F = \ln|x + \sqrt{x^2 - 1}|$ ;

(d)  $F = \ln|\sin^2 x + \ln x| - \ln(\ln \pi)$ .

18. (a)(i)  $a = \pm\sqrt{3}$ ; (ii) any  $b$  except 0 or 3; (b)(i)  $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ ; (ii)  $\mathbf{B}^{-1} = -(1/3) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ ;

(d)(i)  $b=0, c=0, d=\pm 1$ ; (ii)  $b=0, c=0, d=1$ ; (e)  $(1/2) \begin{pmatrix} 2a & b+c \\ b+c & 6 \end{pmatrix} + (1/2) \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$ .

19. (a)(i)  $p > -1$ ; (ii)  $p \geq 0$ ; (b)  $1/(3x^3)$  error  $O(1/x^5)$ .

20. (a) 1; (b)  $\int_{e^{\alpha x}}^{e^{\beta x}} t^2 \cos(xt^2) dt + \sin(xe^{2\beta x})\beta e^{\beta x} - \sin(xe^{2\alpha x})\alpha e^{\alpha x}$ ; equals zero when  $\alpha = \beta$ ;

(c)  $I = (1/2)\ln(\beta/\alpha)$ .

Paper II

A1. (a)  $\lambda = \cosh u + \sinh u : (1 \ 1)^T, \lambda = \cosh u - \sinh u : (1 \ -1)^T$ . 2.  $(-1 \ 1 \ 0)^T$ .

3.  $y = (1/3)e^{2x} + ce^{-x}$ . 4.  $\sqrt{(\pi/a)}$  and zero. 5. (a)  $v = p/m, H = p^2/2m + V(x)$ .

6.  $(-xz \ -yz \ x^2 + y^2)^T, -2z$ . 7.  $c = 1 - e^{-1}, e^{-N}$ . 8.  $(-1)^{n-1} x^{2(n-1)}$ . 9.  $a = -1$ . 10.  $\pi$ .

B11. (a)  $\sqrt{13}$ ; (b)  $1/\sqrt{2}$ ; (c)(i)  $\sqrt{(\mathbf{a}_2 \cdot \mathbf{a}_2 - (\mathbf{a}_2 \cdot \hat{\mathbf{n}}_2)^2)} = |\mathbf{a}_2 - (\mathbf{a}_2 \cdot \hat{\mathbf{n}}_2)\hat{\mathbf{n}}_2|$ ; (ii)  $\sqrt{2/3}$ ;

(d)(i)  $d(t) = \sqrt{(t+1)^2 + 1/2}$ ; (ii) Asymptotes are  $|t+1|$ ; (iii)  $t_{min} = -1$ , when  $d(t) = 1/\sqrt{2}$ .

12. (b)  $f(0,1) = 1, f(0,-1) = -1$ ; (c)  $f=0$  is  $y=x^2$  a parabola,  $f=1$  is  $(0,1)$  a point,

$f=-1$  is  $y=-1 \pm x$ ;  $(0,1)$  is a maximum,  $(0,-1)$  is a saddle point.

13. (a)  $(0 \ 0 \ 2a)^T$ ; (b) zero; (c)  $a=0$ ; (d)  $2a\pi$ ; (e)  $2a\pi$ ; (f)  $\phi = -e^{-r^2}/2$  when  $a=0$ .

14. (a) (ii)  $1/2$ ; (iv)  $2$ ; (b)  $1/4 - \pi/16$ .

15. (a)  $y = x - x/\ln x$ ; (b)(i)  $d^2 v/dt^2 - 4v = 2$ ; (ii)  $v = Ae^{2t} + Be^{-2t} - 1/2$ ; (iii)  $2Ae^{2t} - 2Be^{-2t}$

(c)(i)  $y = 3x^2/4 - 3x/4 + c + de^{-2x}$ ; (ii)  $y = 3x^2/4 - 3x/4 + 11/8 - (3/8)e^{-2x}$ .

16. (a)(i)  $(ay/(x^2+y^2), -ax/(x^2+y^2), 0)$ ; (ii) zero; (iii)  $\psi = a \ln r, \nabla \wedge \mathbf{A} = (0, -a/r, 0)$ ; (b)  $5/2$ .

17. (b)(i)  $\mathbf{L} = \begin{pmatrix} 1 & \lambda & 0 \\ \lambda & 0 & -1 \\ 0 & 4 & 1 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ; (ii) solutions if  $\lambda \neq \pm 2$ ;

(iii)  $x = (3\lambda - 4)/(\lambda^2 - 4), y = (\lambda - 3)/(\lambda^2 - 4), z = (\lambda^2 - 4\lambda + 8)/(\lambda^2 - 4)$ ; (c)(i)  $\begin{pmatrix} a & -2c \\ c & a \end{pmatrix}$ ;

(ii)  $\begin{pmatrix} 0 & \mp 2 \\ \pm 1 & 0 \end{pmatrix}$ .

18. (a)  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ ; (b)  $a_0 = L, a_n = 0$  if  $n$  is even,  $a_n = -4L/n^2 \pi^2$  if  $n$  is odd;

(e)  $\pi^4/96 - 1$ .

19. (a)(i)  $\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$ ; (ii) Saddle point; (b)  $\alpha = 3^{1/4} m, \beta = 3^{-1/4} m$ .

20. (a)  $\Phi = \frac{T_0}{2} - \frac{T_0}{2} e^{-4\pi^2 \kappa t/L^2} \cos \frac{2\pi x}{L}$ ; (b)  $\Phi = \frac{-8T_0}{\pi} \sum_{m=1}^{\infty} \frac{e^{-(2m-1)^2 \pi^2 \kappa t/L^2}}{(2m-1)[(2m-1)^2 - 4]} \sin \frac{(2m-1)\pi x}{L}$ .

## Paper I

A1. (a)  $(a-b)(a^2+b^2+ab)$ ; (b)  $a+b$ . 2.  $x=-1$  or  $3$ . 3.  $x \leq -1$  or  $x > 0$ . 4.  $x=0, y=0$ .

5. (a) 50; (b)  $1/(\sqrt{2}+1)$ . 6.  $x=3, y=27e^{-3}$ . 7. -. 8.  $e^{\sqrt{x}}+c$ . 9.  $(2,0), 1$ . 10.  $2(a^2+a)$ ;  $\pi/3$ .

B11. (a)(i)  $\text{Re}=0, \text{Im}=-e^{-2m\pi}$ ; (ii)  $\text{Re}=\cos\left(\left(\frac{\pi}{2}+2n\pi\right)e^{-\pi/2-2m\pi}\right), \text{Im}=\sin\left(\left(\frac{\pi}{2}+2n\pi\right)e^{-\pi/2-2m\pi}\right)$ ;

(b) circle, centre  $-1-i$ , radius 8; (c)(i) ellipse, centre  $-2-i$ , semiaxes 16 and 8; (ii) ellipse,

centre  $2+2i$ , semiaxes 16 and 8; (iii) ellipse, centre  $-2+2i$ , semiaxes 8 and 16;

(d)  $16\sin^5\theta-20\sin^3\theta+5\sin\theta$ ,  $\theta=0, \pi/6, 5\pi/6, \pi, 7\pi/6, 11\pi/6$ .

12. (a)(i) Limits for  $x$  depend on  $y$ ; (iii) and (iv)  $4/9$ ; (b)  $\pi\rho R^2 h(5.32/15)+\pi R^2\rho d$  (?)

13. (a)  $y=\sqrt{-x^2-2x/3+cx^{-2}}$ ; (ii)  $y=(1-\cos x)/\sin^2 x$ .

14. (a)(i)  $a=1$ ; (ii)  $f=xy^2\sin x+c$ ; (b)(i)  $f_x=2xf_u+2xy^2f_v, f_y=2yf_u+2yx^2f_v$ ; (ii)

$f_{xx}=4x^2f_{uu}+8x^2y^2f_{uv}+4x^2y^4f_{vv}+2f_u+2y^2f_v, f_{yy}=4y^2f_{uu}+8x^2y^2f_{uv}+4x^4y^2f_{vv}+2f_u+2x^2f_v,$

$f_{xy}=4xyf_{uu}+4xy(x^2+y^2)f_{uv}+4x^3y^3f_{vv}+4xyf_v$ ;

(iii) and (iv)  $-\frac{4xy}{(1+x^2+y^2)^2}-\frac{4x^3y^3}{(1+x^2y^2)^2}+\frac{4xy}{(1+x^2y^2)}$ .

15. (a) If  $a=0$  then no Taylor series, if  $a \neq 0$  then  $f=|a|\left(1+\frac{1}{2}(x/a)^2-\frac{1}{8}(x/a)^4+\dots\right)$ ;

(b)(i)  $\ln(1+t(0))+\frac{xt'(0)}{(1+t(0))}$ ; (ii)  $xt'(0)+x^2/2(t''(0)-[t'(0)]^2)$ ;

(iii)  $xt'(0)+x^3/6(t'''(0)-[t'(0)]^3)$ .

16. (a)  $N=6-2e^{-2}$ ; (c) zero; (d)  $(14/3-10e^{-2})/(3-e^{-2})$ ; (e)  $M=2$ .

17. (a)(ii)  $-e^x\cos x+c, e^x\sin x+c$ ; (c)(i)  $1/2$ ; (ii)  $I(n)=\frac{n-1}{2}I(n-2)$ ;

(iii)  $\sqrt{\pi}/4, 1/2, 3\sqrt{\pi}/8, 1, 15\sqrt{\pi}/16$ .

18. (a)  $\text{Tr } \mathbf{A} = 10, \text{Det } \mathbf{A} = 0, \text{Tr } \mathbf{A}^2 = 52, \text{Det } \mathbf{A}^2 = 0$ ; (b)  $\lambda=0, \frac{1}{\sqrt{6}}(1, 1, -2)$ ,

$\lambda=4, \frac{1}{\sqrt{2}}(1, -1, 0), \lambda=6, \frac{1}{\sqrt{3}}(1, 1, 1)$ ; (c)  $\mathbf{x}=k(1, 1, -2)+(2/3, -1/3, 1/6)$ .

19. (a)(i) diverges; (ii) converges to 9; (ii) does not converge; (iv) converges to  $e^2-1$ ; (b)  $\ln 2$ .

20. (a) 1.

Paper II

A1. (a)  $(-4 \ 8 \ -4)^T$ ; (b)  $(0 \ 0 \ 1)^T$ . 2. (a)  $C=AB$ ; (b)  $t=\text{Tr}(A)$ . 3. (a) No; (b) No; (c) Yes.

4. No. 5.  $\lambda=\pm 2$ . 6.  $(1/2)\sin 2x+(1/2)\sin 4x$ . 7. -. 8.  $3x-6x^3$ . 9.  $(x^2/2)\ln x-x^2/4+c$ .

10.  $f_{xx}=2y^3, f_{xy}=f_{yx}=6xy^2, f_{yy}=6x^2y$ .

B11. (a)  $[\mathbf{u}, \mathbf{v}, \mathbf{w}] \neq 0, \alpha = [\mathbf{y}, \mathbf{v}, \mathbf{w}]/[\mathbf{u}, \mathbf{v}, \mathbf{w}], \beta = [\mathbf{y}, \mathbf{w}, \mathbf{u}]/[\mathbf{u}, \mathbf{v}, \mathbf{w}], \gamma = [\mathbf{y}, \mathbf{u}, \mathbf{v}]/[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ ;

(b)(i)  $\mathbf{x} = \mathbf{a} + \lambda \mathbf{c}$ ; (ii) if  $\mathbf{b} \cdot \mathbf{c} \neq 1: \lambda = \mathbf{a} \cdot \mathbf{b} / (1 - \mathbf{b} \cdot \mathbf{c})$  unique value, if  $\mathbf{b} \cdot \mathbf{c} = 1$  and  $\mathbf{a} \cdot \mathbf{b} = 0$ : any  $\lambda$ ,

if  $\mathbf{b} \cdot \mathbf{c} = 1$  and  $\mathbf{a} \cdot \mathbf{b} \neq 0: \lambda$  undefined; (iii) if  $\mathbf{b} \cdot \mathbf{c} \neq 1: \mathbf{x} = \mathbf{a} + (\mathbf{a} \cdot \mathbf{b} / (1 - \mathbf{b} \cdot \mathbf{c})) \mathbf{c}$  a point,

if  $\mathbf{b} \cdot \mathbf{c} = 1$  and  $\mathbf{a} \cdot \mathbf{b} = 0: \mathbf{x} = \mathbf{a} + \lambda \mathbf{c}$  a line.

12. (a)  $\nabla f = (3x^2y + y^3 - y, x^3 + 3xy^2 - x)$ ; (b)  $-3/8\sqrt{5}$ ; (c) saddles:  $(0, 0), (\pm 1, 0), (0, \pm 1)$ ,  
minima:  $\pm(1/2, 1/2)$ , maxima:  $\pm(1/2, -1/2)$ .

13. (a)(i) 5/4; (ii) 11/10; (iii) 5/4; (b) G: no, unless  $\mathbf{a} = \mathbf{0}$ , in which case  $\phi = c$ ,

H: yes,  $\phi = \int r f(r) dr + c$ .

14. (a) (i)  $10y^2(1-y)^3$ ; (ii)  $1 - (1-y)^5 - 5y(1-y)^4 = 1 - (1-y)^4(1+4y)$ ;

(iii)  $1 - [(1-x)^5(1-y)^5 + 5x(1-x)^4(1-y)^5 + 5y(1-y)^4(1-x)^5]$

(b)  $\frac{2x^2(1-y)^2 + 2y^2(1-x)^2}{2x^2(1-y)^2 + 2y^2(1-x)^2 + 5xy(1-x)(1-y)}$ ; (c) 3/100; (d) smallest integer above  $\frac{\ln 0.05}{\ln(1-x)}$ .

15. (a)  $x \sin y = -\frac{1}{2} \cos 2y + c$ , equivalent to  $\sin y = (x \pm \sqrt{x^2 - A})/2$ ; (i)  $y = 0$  or  $y = \sin^{-1} x$

(ii)  $y = \sin^{-1} x$ ; (iii)  $y = \pi - \sin^{-1} x$  where in all three cases,  $\sin^{-1} x$  is defined as an angle between

$-\pi/2$  and  $+\pi/2$ ; (b)  $y(x) = (1/2)(1 - e^{-3x})$ ,  $y(1) = (1/2)(1 - e^{-3})$ .

16. (a) In case it is not clear: the cone is  $(x, y, 1 - \sqrt{x^2 + y^2})$  and the parabola is  $(x, y, -1 + x^2 + y^2)$ ,

and so all of  $x, y, z$  are within  $[-1, +1]$ ; (b)(i) zero; (ii)  $5\pi/6$ .

17. (a) No; (b)(i)  $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & a-1 \\ a & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ; (ii)  $(1-a)(5-2a)$ , vanishes for  $a=1, a=5/2$ ;

(iii)  $a=1, a=5/2$ : no solutions, otherwise one solution.

18. (a)(i)  $\sin nx, \cos nx$  for integer  $n$ ; (ii)  $\frac{1}{\pi} \int_{-\pi}^{\pi} e_n(x) e_m(x) dx = \delta_{nm}$ ; (iii) standard Fourier with

$L = \pi$ ; (b)  $f(x) = \sin x - 2 \sin 2x - \pi^2/6 + \sum_{n=1}^{\infty} (1/n^2) \cos 2nx$ .

19. (b)  $\pm(1/\sqrt{2}, 0, -1/\sqrt{2})$  gives  $T=0$ ,  $\pm(1/2, \sqrt{2}/2, 1/2)$  gives  $T = \sqrt{2}/2$  max,

$\pm(1/2, -\sqrt{2}/2, 1/2)$  gives  $T = -\sqrt{2}/2$  min; (c) add  $\mu(y - h(x))$  to Lagrangian function.

20. (a)  $T = \sum (-2/n\pi)(-1)^n e^{-n^2\pi^2 x/a} \sin n\pi y$ ; (b)  $a=0, b=1$  or  $a=1/6, b=3/4$ ,

$h = -t$  or  $h = (1/(6t^{1/3}))(1 - x^2/t^{2/3}) - 3t/4$ .

## Paper I

A1. 70, 56. 2.  $(1 \pm \sqrt{5})/2$ . 3. (0,1) and (3, -2). 4. (-1, -2) radius 3. 5.  $\phi = 30^\circ, A = 2$ .

6.  $p = q = 31$ . 7.  $x = -1, f = 1/\sqrt{e}; x = 1, f = \sqrt{e}$ . 8.  $dy/dx = -(1+e^x)/(1+e^y)$ . 9.  $n = 2$ .

10.  $2 - 2 \ln 2 - (\ln 2)^2$ .

B11. (a)(i)  $\text{Re} = 18, \text{Im} = 1$ ; (ii)  $\text{Re} = 6/25, \text{Im} = -17/25$ ; (b)(i)  $y = x(\sqrt{2}-1)$  or  $y = -x(\sqrt{2}+1)$ ;

(ii)  $y = \pm x$ ; (c)  $x = -\text{arcosh}(2/\sqrt{3}) = -\ln \sqrt{3}$ ; (d)  $z = \ln(1+\sqrt{2}) + i(\pi/2 + 2n\pi)$  or

$z = \ln(\sqrt{2}-1) + i(-\pi/2 + 2n\pi)$ .

12. (a)(i)  $V = \pi R^2 H(3 - e^{-2})/8$ ; (ii)  $a = \sqrt[3]{3R^2 H(3 - e^{-2})/50}$ ; (b)  $V = 2 DLH(2 + \sqrt{2})/15$  (I

originally got the same as the official answer, which is  $V = 4 DLH(2 + \sqrt{2})/15$ , but now think that is wrong by the factor of 2).

13. (a)  $y = (4/x + \ln|x| + c)^{-1}$ ; (b)  $y = \tan x + \sqrt{2} \sec x$ ; (c)(i)  $y = x((-1)^n \sin^{-1}(Ax) + n\pi)$ ;

(ii)  $y = x \sin^{-1}(\sqrt{3}x/2)$ ; (iii)  $y = x(\pi - \sin^{-1}(\sqrt{3}x/2))$

14. (a)  $V = \sqrt{x}(1 + \sqrt{1+y^2}) + c$ ; (b)  $dV/V = 2dr/r + dh/h$ ,  $dV/V = 0.04$ .

15. (a)  $x^3 + x^7/3! + x^{11}/5!$ ; (b)  $x - x^2 + 7x^3/6$ ; (c)  $x^6 - x^8 + 7x^{10}/15$ .

16. (a)  $A = 12$ ; (c)  $f(x) = 12(x^2 - x^3)$  for  $0 \leq x \leq 1$ , zero otherwise; (e) Mode is  $2/3$ ;

(f) Median < Mode; (g)  $E(X) = 3/5, \sigma = 1/5$ ;

(h)  $12((4/5)^3/3 - (4/5)^4/4 - (2/5)^3/3 - (2/5)^4/4) = 0.64$ .

17. (c)  $a_n = \frac{4}{n^2 \pi^2} (-1)^n, a_0 = 2/3, b_n = 0$ ; (d)  $b_n = \frac{2}{n\pi} (-1)^{n+1}, a_n = 0$ ; (e)  $A = 4, B = 90$ .

18. (a)  $\beta = 1, \gamma = 1 - \alpha$ ; (b)  $A_\alpha = \begin{pmatrix} 1+\alpha & 1-\alpha & 2 \\ 2 & \alpha & 2-\alpha \\ 2-\alpha & \alpha+3 & 1 \end{pmatrix}$ ; (c)  $10\alpha^2 - 10\alpha + 8$ .

19. (b)(i) converges; (ii) converges if  $a < 1$ ; (d)(i) zero; (ii)  $a^a(\ln a + 1)$ ;

(iii)  $0 \leq a \leq 1: 1, 1 < a \leq 2: a, a > 2: a^2/2$ .

20. (b)(i) Minimise  $\frac{1}{c} \left( \frac{2d_1}{\cos \theta} + \frac{\sqrt{2}d_2}{\cos \phi} \right)$  subject to  $d_1 \tan \theta + d_2 \tan \phi = D$ ,  $\sin \phi = \sqrt{2} \sin \theta$ ;

(ii)  $\frac{\sqrt{3} \sin \theta}{\sqrt{(1 - \sin^2 \theta)}} + \frac{\sqrt{2} \sin \theta}{\sqrt{(1 - 2 \sin^2 \theta)}} = 2$ ,  $\sin \theta = 1/2$ ; (iii)  $\theta = 30^\circ, \phi = 45^\circ$ , time taken =  $6/c$ ,

distance =  $2 + \sqrt{2}$ .

Paper II

- A1. (a)  $X=A$  ; (b)  $Y=\Omega$  . 2. (a)  $r=5, \theta=\arctan(4/3)+\pi, z=-1$  ;  
 (b)  $r=\sqrt{26}, \theta=\pi-\arctan(5), \phi=\arctan(4/3)+\pi$  . 3. (a)  $\alpha=0, \beta=\mathbf{a} \cdot \mathbf{c}, \gamma=-\mathbf{a} \cdot \mathbf{b}$  ;  
 (b) Area of parallelogram formed by  $\mathbf{a}$  and  $\mathbf{b}$  . 4.  $(1, -1, 0)/2$  . 5. Cumulative normal distribution.  
 6. (a)  $x_1=x_0-f(x_0)/f'(x_0)$  ; (b)  $3/2$  . 7. (a)  $(-1, -1, -1)$  ; (b)  $-\sqrt{2}$  . 8. (a) zero; (b) zero.  
 9. (a)  $(\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b})$  ; (b)  $3(\mathbf{b} \cdot \mathbf{b}+1)$  . 10. (a)  $z=-x-y$  any  $x, y$  ; (b) Plane  $x+y+z=0$  .  
 B11. (c)(i) No; (ii) Yes.  
 12. (a)  $x=\ln(c/a), y=\ln(d/b), h=c+d-c \ln(c/a)-d \ln(d/b)$  ;  
 (b)  $c>0, d>0$ : min,  $c$  and  $d$  different signs: saddle,  $c<0, d<0$ : max ; (c)(i) Min at  $(0, 0)$  ;  
 (ii) Saddle at  $(0, 0)$  .  
 13. (a)(i)  $F: a, G: \sin(1)+c$  ; (ii)  $F: a, G: \sin(1)+c$  ; (b)  $b=0$ , any  $a, c$  .  
 14. (a)(i)  $6!$  ; (ii)  $5!$  ; (iii)  $2(5!)=240$  ; (iv)  $4!$  ; (v)  $2(5!)-4!=216$  ; (vi)  $2(5!)-2(4!)$  ; (b)(i) one;  
 (ii)  $(4!)(2!)$  ; (iii)  ${}^6C_4$  ; (v) For  $\{m, 6-m\}$ , (ii) is  $m!(6-m)!$ , (iii) is  ${}^6C_m$  ; (iv) is  $6!$  .  
 15. (a) if  $k=1$   $y=A \cos x+B \sin x+(x/2) \sin x$ , if  $k \neq 1$   $y=A \cos x+B \sin x+(\cos kx)/(1-k^2)$  ;  
 (b)(i)  $\alpha=a+d, \beta=ad-bc$  ; (ii)  $u=-te^{2t}, v=(1-t)e^{2t}$  .  
 16. (a)  $3+e^{-1}$  ; (b)(i)  $dS=\sqrt{2} dx dz$  ; (ii)  $\sqrt{2} \pi(2\pi+5/3)$  ; (c)(i)  $4\pi$  ; (ii) zero.  
 17. (b)  $Q=\begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$  ; (c)(i)  $\frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$  ; (ii)  $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$  ; (iv)  $\begin{pmatrix} 1366 & 2730 \\ 1365 & 2731 \end{pmatrix}$  .  
 18. (a)(i)  $\int \sin^6 x dx = \frac{5x}{16} - \cos x \left( \frac{1}{6} \sin^5 x + \frac{5}{24} \sin^3 x + \frac{5}{16} \sin x \right) + c_1$  ,  
 $\int \cos^6 x dx = \frac{5x}{16} + \sin x \left( \frac{1}{6} \cos^5 x + \frac{5}{24} \cos^3 x + \frac{5}{16} \cos x \right) + c_2$  ; (b)  $\pi/4, 3\pi/16, \pi/4, 3\pi/16$  .  
 19. (a) Equality when  $f=-\lambda g$  ; (b)(i)  $K=\pi h \int_{r=0}^a r^3 \omega^2(r) dr$  ; (ii)  $L=2\pi h \int_{r=0}^a r^3 \omega(r) dr$  ,  
 $f=\sqrt{(\pi h r^3)} \omega(r), g=2\sqrt{(\pi h r^3)}$  ; (iii)  $\omega(r)=\text{constant}$  .  
 20. (a)  $u=A e^{kx} (2y-6)^{k/2}$  ; (b)  $u=A y^k \exp(-k(x+1)e^{-x})$  ; (c)  $u=e^{-2\cos x} \sin^2 y$  .

## Paper I

A1. (a)  $\sqrt{6}$ ; (b)  $150^\circ$ . 2.  $y=3x-2$ . 3.  $-2xe^{2-x^2}$ ,  $f=e^2$ . 4. (a) 780; (b) 6.

5.  $3/2 - \sqrt{5}/2 \leq x \leq 3/2 + \sqrt{5}/2$ . 6. (a) 2; (b) (1,0) and (0,1). 7.  $x=30^\circ$ .

8. radius=2, centre=(1, -1/2). 9.  $n=-2, a=1$ . 10. (a)  $x \sin x + \cos x + c$ ; (b) 2.

B11. (d)(i) dot product; (ii) area of parallelogram; (f) N.

12. (b)  $2\pi\gamma\rho a^3(\beta^2 - (\alpha - \gamma)^2)$ ; (c)  $\frac{4}{3}\pi\rho\beta^3 a^3 - \pi\rho a^3\left(\frac{2}{3}\beta^3 - \beta^2(\alpha - \gamma) + (\alpha - \gamma)^3/3\right)$ , ie  $\pi\rho a^3\left(\frac{2}{3}\beta^3 + \beta^2(\alpha - \gamma) - (\alpha - \gamma)^3/3\right)$ ; (d) as (c); (e) it's (b)+2(c) obviously.

13. (a)  $y=(4/x+x^3/3+c)^{-1}$ ; (i)  $y=(4/x+x^3/3-10/3)^{-1}$ ; (ii)  $y=0$ ; (b)(i)  $\mu = \frac{1}{\sqrt{16+x^2}}$ ;

(ii)  $y = \frac{16+x^2}{3} + \frac{c}{\sqrt{16+x^2}}$ .

14. (a)  $dc/c = \frac{ada+bd b}{a^2+b^2}$ ,  $dA/A = da/a + db/b$ , -0.01, so 1% decrease.

15. (a)  $\ln(1.1) \approx 0.095$ , upper bound  $1/3000$ ; (c)(i)  $1 - x^2/4 + 7x^4/96$ ;

(ii)  $e - (e/2)(x - \pi/2)^2 + (e/6)(x - \pi/2)^4$ .

16. (a)(ii)  $1 - 2p_w$ ; (b)(i)  $2x - 2p_w$ ; (ii)  $\frac{1-2x+p_w}{1-2x+2p_w}$ ; (c)(i)  $N=20$ ; (ii)  $\binom{10}{4}\binom{10}{8}/\binom{20}{12}$ .

17. (a)  $2+2\ln 2$ ; (b)  $1/2 - 1/(1+\sqrt{3})$ ; (c)  $(1/2)\ln(e^{2x}+1) - 2\arctan(e^x) + c$ ;

(d)  $(1/2)\arctan\left(\frac{\tan x}{2}\right) + c$ .

18. (a)(i)  $\lambda=-2, \mathbf{v}=(0 \ 1 \ -1)^T/\sqrt{2}$ ,  $\lambda=4, \mathbf{v}=(1 \ 0 \ 0)^T$ ,  $\lambda=6, \mathbf{v}=(0 \ 1 \ 1)^T/\sqrt{2}$ ;

(ii)  $(1/2)\begin{pmatrix} 2 & -3 & 1 \\ 0 & -1 & 1 \\ -2 & 7 & -3 \end{pmatrix} \mathbf{x} = (1/2)(-1 \ 1 \ 3)^T$ ; (b)(i) reflection in one of the xy-plane, the xz-plane, or the yz-plane.

19. (a)  $e^x(x^2+200x+9900)$ ; (b)(ii)  $p>0$ ; (iii)  $p>1, f'(0)=0$  (c)(i) converges; (ii) diverges.

20. (b)(i) and (ii)  $6\pi^2 x^5 - 4x^3 + \cos x - \cos \pi x^2 + 2\pi x^2 \sin \pi x^2 - x \sin x$ ; (c)(i)  $\frac{dg}{dx} = \frac{g}{x+1}$ ;

(iii)  $g=x+1$ ; (iv) 1.

Paper II

A1. (a) 1; (b) all zero. 2. (a)  $r = \sqrt{2}$ ; (b)  $\theta = \pi - \arctan(\sqrt{2})$ . 3.  $\xi = 1/\ln 2$ . 4. (a)  $f(x)$ ; (b)  $\cos x(1 - \cos x) + \sin^2 x$ . 5.  $\alpha = -1, \beta = -1$ . 6. (a) zero; (b)  $P(A) + P(B)$ . 7.  $f = e^{xy} - 1$ . 8. (a) 1; (b) zero. 9.  $a_{21} = 4, a_{22} = 4$ . 10.  $(1/2)\sin 2x + \sin x$ .

B11. (a)  $R_{\hat{u}}(\mathbf{r}) = \mathbf{r} - P_{\hat{u}}(\mathbf{r})$ ,  $P_{\hat{u}}(\mathbf{r})$  and  $R_{\hat{u}}(\mathbf{r})$  are two sides of a right-angled triangle which has  $\mathbf{r}$  as its hypotenuse, with  $P_{\hat{u}}(\mathbf{r})$  parallel to  $\hat{u}$ ; (b)  $\mathbf{r}$ ; (c) eg  $\mathbf{p} = \mathbf{y}, \hat{\mathbf{q}} = \frac{\mathbf{z} - \mathbf{y}}{|\mathbf{z} - \mathbf{y}|}$  or equivalent; (d)(ii)  $\hat{\mathbf{a}}$  is parallel to the line; (iii)  $c$  = distance of line from origin; (iv) the line and the origin; (e)  $\mathbf{r} = \hat{\mathbf{g}} \times \mathbf{f} + t \hat{\mathbf{g}}$ .

12. (a)  $(6x^5 - 48xy^2, 6y^5 - 48x^2y)$ ; (b)  $y=0$  or  $x = \frac{\pm 1}{\sqrt{8}}y^2$ ,  $x=0$  or  $y = \frac{\pm 1}{\sqrt{8}}x^2$ ;

(c)  $(0,0), (\pm\sqrt{8}, \pm\sqrt{8})$ ; (d) minima; (e) saddle.

13. (a) no for  $\mathbf{G}$ , yes for  $\mathbf{K}$ ,  $\phi = (1/2)\ln(1+r^2) + c$ ; (b)(i)  $(4+\mu+\lambda)/3$ ;

(ii)  $\lambda + 5/3$ ;  $\mu = 2, \lambda = 1/2$ .

14. (a)(ii) mean =  $1/2$ , variance =  $1/12$ ; (b)(iii) mean =  $0$ , stdev =  $1/\sqrt{6}$ ; (iv)  $1/8$ .

15. (a)  $y=4$ ; (ii)  $y = Ae^{-x} + Be^{-5x}$ ; (iii)  $y = e^{-3x/2} \left( A \cos \frac{\sqrt{7}x}{2} + \sin \frac{\sqrt{7}x}{2} \right)$ ;

(b)  $y = 25x^2 + 40x + 22 - 21e^{2x} \cos x$ .

16. (a)  $(2yx + 4x^2y^2z^2 + x^4z^2 + x^4y^2)e^{x^2yz}$ ; (b)(ii)  $OAC : -1/4, OBC : -1/3, ABC : 1$ ; (iii)  $1/4$ .

17. (b)  $3(2\alpha+1)(\alpha-3)$ , zero when  $\alpha=3$  or  $\alpha=-1/2$ , any  $\beta$ ;

(c)(i)  $x = (1-2\alpha)/(1+2\alpha), y = 2/(1+2\alpha), z = 0$ ;

(ii)  $x = -5/7 - (18+\beta)z/7, y = 2/7 - (3-\beta)z/7$ , any  $z$ ; (iii) no solution.

18. (c)(i)  $f(x) = \sum_{\text{odd } n} \frac{8}{n^3 \pi^3} \sin n\pi x$ ; (ii)  $A = 1/960$ .

19. (a)  $r = \sqrt{2/3}R, h = 2R/\sqrt{3}$ , fraction is  $1/\sqrt{3}$ ; (b)(i)  $\max f = 1$  at  $\pm \left( \frac{1}{\sqrt{8}}, \sqrt{8} \right)$ ,

$\min f = -1$  at  $\pm \left( \frac{1}{\sqrt{8}}, -\sqrt{8} \right)$ .

20. (b)  $\psi = 1/2 + (1/2)\cos x \cos vt$ ; (c)(i) same as (b).