<u>2025</u>

Paper I

A1. (a) $x=1$. 2. $m=2, k=2$. 3. (a) $a=0$; (b) π. 4. (a) $a=1$.

5. (a) $(1/4)\sin 2x - x/2 + c$; (b) $-2/\sqrt{3}$. 6. (a) 25; (b) 1/18. 7. (a) $1-e$; (b) 1.

8. $a=1, b=-2, c=5$. 9. (a) $(0,1)$; (b) 2. 10. (a) 1/12; (b) 5/18.

B11. (a)(i) plane, includes origin; (ii) $(1,1,1)^T$ and $(2,-1,0)^T$; (iii) No; (b)(i) $(x-2)^2/2 + y^2 = 4$; (ii) cylinder; (iii) circle; (iv) $\sqrt{2}$; (c)(i) $|(\boldsymbol{a_1}-\boldsymbol{a_2}).(\boldsymbol{v_1}\times\boldsymbol{v_2})/|(\boldsymbol{v_1}\times\boldsymbol{v_2})||$; (ii) $|(\boldsymbol{a_1}-\boldsymbol{a_2})\times\boldsymbol{v_1}/|\boldsymbol{v_1}||$; (iii) $(\boldsymbol{a_1}-\boldsymbol{a_2}).(\boldsymbol{v_1}\times\boldsymbol{v_2})=0$.

12. (a)(i) $x - x^3/3$; (ii) $(2\ln 2)x^2 + (1/3)((\ln 2)^3 - \ln 2)x^4$;

(b) if $e^a - b \neq 1$ $\ln(e^a - b) + ae^a(x-1)/(e^a-b)$; if $e^a - b = 1$ $ae^a(x-1) + (1/2)a^2e^a(1-e^a)(x-1)^2$.

13. (a) $10e^{1/2} - 16$; (b) zero; (c) $\ln|x| - \frac{1}{2}\ln(x^2+1) + c$; (d) $\sinh^{-1}x + \sqrt{1+x^2} + c$;

(e) $(1/2)\tan^{-1}((1/2)\tan x/2) + c$.

14. (a) $|a| \leq 3\sqrt{3}/\pi$; (b) $a=1$; (c) $\pi^2/36 + \sqrt{3}\pi/3 - 2$; (d) $\pi/6 - (2a/3)(\pi/6)^3$.

15. (a)(i) $y = C\sec x$; (ii) $y^2 = 2x^2\ln|x| + cx^2$; (b) yes, $u = [-1 \pm \sqrt{(1+4x^2-4x^4)}]/2x^2$.

16. (a) $(1,-1),(2,0),(2,-2),(0,0),(0,-2)$;

(b) $(1,-1): f=0, (2,0): f=e^{-1}, (2,-2): f=-e^{-1}, (0,0): f=-e^{-1}, (0,-2): f=e^{-1}$;

(c) $(1,-1): saddle, (2,0): max, (2,-2): min, (0,0): min, (0,-2): max$.

17. (b)(i) $\begin{pmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{pmatrix}$; (ii) $n_1^2 + n_2^2$; (c) vectors are mapped onto their projection perpendicular to $\hat{\boldsymbol{n}}$; (d) 2; (e) reflection in plane perpendicular to $\boldsymbol{p}$;

(f) reflection through z-axis; $\boldsymbol{y} = (-x_1, -x_2, x_3)$; (g) $\boldsymbol{r} = (0, 0, \sqrt{2})$.

18. (a)(i) $e^\pi(-1)^n - 1 + n\int_0^\pi e^x \sin nx\, dx$; (ii) $-n\int_0^\pi e^x \cos nx\, dx$; (iii) $\int_0^\pi e^x \cos nx\, dx = \frac{(-1)^n e^\pi - 1}{n^2+1}$; $\int_0^\pi e^x \sin nx\, dx = \frac{n(1-(-1)^n e^\pi)}{n^2+1}$; (c) $\frac{1}{\pi}(e^\pi - 1) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n e^\pi - 1}{n^2+1}\cos nx$; (d) $m=4$.

19. (b) In case it confuses you too, they mean $\lim_{n\to\infty} u_n = 0$; (e) $m=2$, I think what you get for A depends on your method - I got $A=1$; (f) it does converge.

20. (a) $T=0$ at $(0,\pm a,0)$ and $(0,0,\pm b)$; (b) $T = \frac{a^4b^2}{b^2+4a^2}$ at $\left(\frac{\pm ab}{\sqrt{b^2+4a^2}}, \pm\frac{\sqrt{2}a^2}{\sqrt{b^2+4a^2}}, \pm\frac{\sqrt{2}ab}{\sqrt{b^2+4a^2}}\right)$, $T = a^2b^2/4$ at $(0, \pm a/\sqrt{2}, \pm b/\sqrt{2})$, $T = a^4$ at $(\pm a, 0, 0)$; (c) $T=1$ at $(\pm 1, 0, 0)$.

Paper II

A1. (a) $(1/2)(-1,-1,-1)$; (b) $-1/2$. 2. no. 3. (a) $-2b+3=0$; (b) $b=3/2$.

4. $(1/2)(\cos 8x+\cos 2x)$. 5. (a) $\boldsymbol{a}.\boldsymbol{b}$; (b) $\mathrm{Tr}(\boldsymbol{C}\boldsymbol{D}^{\boldsymbol{T}})=\mathrm{Tr}(\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{D})$. 6. $y=Ae^{-3x}+Be^{2x}$.

7. $\pi^2\rho_0/4$. 8. (a) y^2; (b) $(0,0,-2xy)$. 9. (a) $\lambda^2-3\lambda+1=0$; (b) $(3\pm\sqrt{5})/2$.

10. (a) 1/8; (b) 3/8.

B11. (a)(i) a straight line; (ii) $2b_1x+2b_2y+c=0$; (b)(i) circle, centre $-b$ and radius $\sqrt{b\bar{b}-c}$;

(ii) $(x+b_1)^2+(y+b_2)^2=b_1^2+b_2^2-c$; (iii) $b\bar{b}-c\geq 0$; (iv) $b\bar{b}-c=0$; (c)(i) $A=a, B=b\bar{\alpha}, C=c$

(ii) $A=a\alpha^2, B=b\alpha, C=c$ where α is real; (iii) $A=a, B=a\alpha+b, C=a|\alpha|^2+2\,\mathrm{Re}(b\bar{\alpha})+c$;

(iv) $A=c, B=\bar{b}, C=a$, $c=0$.

12. (a) $[g(g-1)(g-2)]/[(g+r)(g+r-1)(g+r-2)]$; (b)(i) zero; (ii) $2gr/[(g+r)(g+r-1)]$;

(iii) $[2g(g-1)r(r-1)]/[(g+r)(g+r-1)(g+r-2)(g+r-3)]$; (c)(i) 2; (ii) 4;

(d) the official answer is that for $x\leq g: 2x/(n+r)$, and for $x>g:(g+x)/(g+r)$; however, I did not feel that distinction was appropriate, and got $[1+(3g+r)/(g+r)^2]/2$, and I've verified that with a simulation; (e)(i) 9/19; (ii) 9/19; (iii) 1/2.

13. (a)(i) $d^2y/dt^2=\beta y$; (ii) for $\beta>0: y=Ae^{\sqrt{(\beta)}t}+Be^{-\sqrt{(\beta)}t}$,

for $\beta<0: y=A\cos\sqrt{(-\beta)}t+B\sin\sqrt{(-\beta)}t$, for $\beta=0: y=At+B$; (b) $y=(t^3/2+3t^2/2-3t+1)e^{3t}$.

14. (a)(i) $(\partial u/\partial\xi)_\eta+(\partial u/\partial\eta)_\xi$; (ii) $(1/\sqrt{y})(\partial u/\partial\xi)_\eta-(1/\sqrt{y})(\partial u/\partial\eta)_\xi$;

(iii) $\partial^2u/\partial\xi^2+2\partial^2u/\partial\eta\partial\xi+\partial^2u/\partial\eta^2$;

(iv) $(1/2)y^{-3/2}(\partial u/\partial\eta-\partial u/\partial\xi)+y^{-1}\partial^2u/\partial\xi^2-2y^{-1}\partial^2u/\partial\eta\partial\xi+y^{-1}\partial^2u/\partial\eta^2$;

(b) $\partial^2u/\partial\eta\partial\xi=0$.

15. (a) zero; (b)(i) 1/18; (ii) $19e^{-1/2}+e-14$.

16. (a)(i) -1/2; (ii) $\phi=y^2x-x^2-y^2/2+c$; (b) $(R_{yx}-Q_{zx})(x')^2+(P_{zy}-R_{xy})(y')^2+(Q_{xz}-P_{yz})(z')^2+$

$(R_{yy}-Q_{zy}+P_{zx}-R_{xx})x'y'+(R_{yz}-Q_{zz}+Q_{xx}-P_{yx})x'z'+$

$(P_{zz}-R_{xz}+Q_{xy}-P_{yy})y'z'+(R_y-Q_z)x''+(P_z-R_x)y''+(Q_x-P_y)z''$.

17. (a)(i) $2\pi(1+z_0)+z_0\pi(1-z_0^2)$; (b)(i) $2\pi(f(z_0)-f(-1))+f'(z_0)z_0\pi(1-z_0^2)$; (ii) $f(z)=z$.

18. (a) $\boldsymbol{S}=(1/2)(\boldsymbol{M}+\boldsymbol{M}^{\boldsymbol{T}}), \boldsymbol{A}=(1/2)(\boldsymbol{M}-\boldsymbol{M}^{\boldsymbol{T}})$; (b)(i) $\alpha_i=(\boldsymbol{x}.\boldsymbol{u}_i)/|\boldsymbol{u}_i|^2$, $\beta_i=\lambda_i(\boldsymbol{x}.\boldsymbol{u}_i)/|\boldsymbol{u}_i|^2$;

(ii) eigenvectors of $\boldsymbol{C}$ are the $\boldsymbol{u}_i$, with eigenvalues e^{λ_i}; (iii) eigenvectors are the $\boldsymbol{u}_i$, with eigenvalues $1/\lambda_i$; (iv) $(1,0)=(1/2)(1,-1)+(1/2)(1,1)$, $\boldsymbol{S}^{-1}\boldsymbol{x}=-(1/2)(1,-1)+(1/6)(1,1)$;

(c)(i) geometric action is scaling by a, $\boldsymbol{A}^2$ is inversion in origin and scaling by a^2; (ii) $\theta=a$.

19. (b) $f_0(x_n)=1, f_0(x'_n)=0$; (c)(i) $f_0(x)$ is continuous if $a>0$ and not otherwise;

(ii) $f_0(x)$ is differentiable if $a>1$ and not otherwise; (d) convergent (like $p=2$ series).

20. (b) $f(x)=(1/2)[u(x)+(1/c)\int_0^x v(y)dy+\text{const.}]$, $g(x)=(1/2)[u(x)-(1/c)\int_0^x v(y)dy-\text{const.}]$;

(c) $\psi=e^{-(x-ct)^2/2}-e^{-(x+ct)^2/2}$.