UNIVERSITY OF CAMBRIDGE

ECONOMICS TRIPOS PART I

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Friday 14 June    9:00-12:00

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Paper 3

QUANTITATIVE METHODS IN ECONOMICS

There will be a 15 minute reading time prior to the beginning of the examination.

This paper is divided into four Sections:
Sections A and B are Mathematics; Sections C and D are Statistics.

You should do the appropriate number of questions from each Section.
The number of questions to be attempted is at the beginning of each Section.
Answer all parts of the question.

Answers from the Mathematics and the Statistics Sections must be written in separate booklets with the letter of the Section written on each cover sheet.

Section A carries 30% of the marks
Section B carries 20% of the marks
Section C carries 30% of the marks
Section D carries 20% of the marks

Write legibly.

| STATIONERY REQUIREMENTS | SPECIAL REQUIREMENTS |
| --- | --- |
| 20 Page booklet x 2 | New Cambridge Elementary Statistical Tables |
| Metric graph paper | Approved calculators allowed |
| Rough work pads | |
| Tags | |

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

**SECTION A - Answer ALL FOUR questions from this Section.**

1 Determine whether the functions below are concave or convex on the specified intervals.

(a) $f(x) = -3\ln(x\sqrt{x})$, for $x > 0$.

(b) $f(x,y) = x^4 - 6x^3 + 9x^2 + y^2 + 6y + 9$, for all $x$ and $y$.

2 An agent has the utility function

$$U(c_1, c_2) = 2 + 3\left(c_1^{\frac{1}{3}} - 1\right) + 3\beta\left(c_2^{\frac{1}{3}} - 1\right)$$

where $c_1 \geq 0$ and $c_2 \geq 0$ is consumption in periods 1 and 2. Finally, $\beta \in (0,1)$ is the factor with which the agent discounts the future.

(a) Solve the agent's utility maximisation problem subject to the constraint that $c_1 + c_2 = 1$, i.e. solve for the optimal values of $c_1$ and $c_2$.

(b) Assume that the agent lives for two periods, but that in addition to his utility above, he also derives utility from leaving a bequest $z$. In other words, that

$$U(c_1, c_2, z) = 2 + 3\left(c_1^{\frac{1}{3}} - 1\right) + 3\beta\left(c_2^{\frac{1}{3}} - 1\right) + z$$

Also, assume that $\beta = 0.5$ and that utility is maximised subject to the constraint $c_1 + c_2 + z = 1$, and that $c_1, c_2, z \geq 0$. Solve the agent's utility maximisation problem, i.e. for the optimal values of $c_1, c_2$ and $z$.

3 Assume that

$$f(x,y) = 2x^2 + 4xy + \frac{1}{12}y^4$$

(a) Find the Hessian matrix for this function of two variables. Calculate the determinant.

(b) Find the stationary points and classify them.

4 Give a Taylor series approximation of the following functions.

(a) $f(x) = e^{3x}$ at $x_0 = 0$ to the second order.

(b) $g(x) = \ln x$ at $x_0 = 1$ to the minumum order such that the approximation is accurate to one decimal place in the interval $[0.5, 1.5]$.

(c) A tangent-plane approximation of $h(x,y) = (x^2 + 7y^3)^{\frac{1}{2}}$ at $(x_0, y_0) = (3, 1)$.

**SECTION B - Answer ONE question from this Section.**

5 There are two otherwise identical firms, one domestic firm $D$ and one foreign firm $F$, selling the same good. Both firms have the same cost function

$$c(q_i) = \alpha q_i$$

where $\alpha > 0$ and $q_i$ is the quantity of the good produced by firm $i = D, F$. Initially there is no trade and firms only operate in their respective home markets where they face the following demands

$$q_D = 25 - p_D \geq 0$$

$$q_F = 15 - p_F \geq 0$$

and prices $p_i$ are positive. Profits are given by

$$\pi_i(q_i, p_i) = p_i q_i - c(q_i)$$

and if the profit associated with producing a certain quantity is negative, then a firm will chose to produce nothing.

(a) Assume that free trade is introduced and that there are no costs associated with trading. Write down the demand function for the new 'global' market where prices are the same in both countries, i.e. write down $Q = q_D + q_F$.

(b) Assume that now that free trade has been introduced, the two firms decide to collude instead of competing with each other. More specifically, assume that they act as if they were one firm. Set up and solve their joint profit maximisation problem. That is, express the optimal quantities, prices and profits as functions of $\alpha$. Be careful to specify for what values of $\alpha$ your results hold.

(c) In part (b), how does a change in $\alpha$ affect $\Pi$? Use the envelope theorem and assume that $\alpha = 10$. Comment.

(d) Assume again that the firms collude like in part (b) and that $\alpha = 10$. Politicians in the 'foreign' country argue that free trade hurts their consumers and request that the quantity of the good sold in their country should be at least 4, in other words that $q_F \geq 4$. Solve the joint profit maximisation problem, and calculate the optimal values of $p$, $Q$ and $\Pi(Q, p)$ if this suggestion is implemented. Compare your answer to what you found in part (b).

6 Goodwin Bank's current accounts only allow holders to make withdrawals and deposits on 1st January each year. Interest is paid into the account annually on 1st January, and calculated at 5% of the account's balance on 2nd January the previous year.

Geraldine opened an account on 1st January 2000. She deposited £10 on the same day she opened the account and then deposits a further £10 each subsequent year, continuing indefinitely. She has never withdrawn and will never withdraw any funds.

(a) Writing $x_t$ for her balance in pounds on 2nd January of the year $t \geq 2000$, express $x_t$ in terms of $x_{t-1}$.

(b) Allow $x_t$ to be defined for $t = 1, 2, \ldots$ by the difference equation you wrote down for part (a). Using what you know about Geraldine's bank balance from 2nd January 1999 onwards, solve the difference equation.

Aisha also has a current account with Goodwin Bank. On 1st January each year, her aunt transfers £10 as a New Year's gift, and Aisha withdraws an amount equal to her balance the next day minus three quarters of her balance on 2nd January the previous year (if this is a negative number, she makes a deposit).

(c) Let $y_t$ be Aisha's balance on 2nd January of year $t$ and $e_t$ be the amount she withdraws on 1st January of year $t$. Obtain a difference equation for $y_t$ of the form $y_t = ay_{t-1} + b$ where $a$ and $b$ are constant parameters. Find the steady-state value of $y_t$, and determine whether $y_t$ converges.

Suppose now that Aisha's aunt, who wants to encourage her niece to save, deducts whatever Aisha withdraws from (or adds whatever she deposits to) her annual £10 gift. That is, she pays $f_t = 10 - e_t$ into Aisha's account in year $t$, and we allow $f_t$ to be negative.

(d) Obtain a new difference equation for Aisha's balance, find its steady-state values, and determine whether it converges.

(e) Briefly explain why Aisha's aunt's attempt to induce higher saving fails.

**Section C - Answer ALL FOUR questions from this Section**

7 Every year there are twice as many male professional tennis players as female players, and 30% of male and 60% of female players get a sore throat (due to fatigue or simply overusing their voice).

(a) What percentage of professional tennis players gets a sore throat each year?

(b) If you get to meet one randomly chosen professional tennis player at a charity event at the end of the year, what is the probability you meet a female player or a player who did not suffer a sore throat?

(c) Suppose you are told that a randomly selected professional tennis player has a sore throat. What is the probability that the player concerned is male?

8 Consider a continuous random variable $X$ that has density

$$f(x) = \begin{cases} x^2 + c & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

(a) Find the value of $c$ and sketch $f$.

(b) Use your answer from part (a) to calculate $E(X - c)$ and $Var(X - c)$.

9 Suppose $X$ and $Y$ are discrete random variables with joint probabilities specified by the following table:

| $P(X = x, Y = y)$ | | $x$ | | |
|---|---|---|---|---|
| | | -1 | 0 | 1 |
| $y$ | -1 | 0.30 | 0.10 | 0.10 |
| | 1 | 0.10 | 0.10 | 0.30 |

(a) Calculate $E(Y)$ and $E(Y|X = 0)$.

(b) Are $X$ and $Y$ independent? No marks will be awarded without justification of answers.

10 For some sequence of numbers $\{x_i\}_{i=1}^n$, you observe $\{y_i\}_{i=1}^n$ that is generated by

$$y_i = \beta\sqrt{x_i} + \varepsilon_i \quad \text{for} \quad i = 1, \ldots, n$$

for some unknown $\beta$ and an unobserved random sample $\{\varepsilon_i\}_{i=1}^n$ that has zero mean and variance $\sigma^2$.

(a) Derive the least squares estimator for $\beta$.

(b) Show that your estimator in part (a) is unbiased.

**Section D - Answer ONE question from this Section.**

11 An economic consulting firm, named Zeta, only hires graduates from three Colleges, A, B or C. The bonus each graduate earns depends on the College s/he graduated from. In particular, the bonuses are independently drawn from a normal distribution that has standard deviation 1000 and mean $\mu_A, \mu_B$ or $\mu_C$ corresponding to College A,B or C respectively. Suppose $\mu_A = 7500, \mu_B = 7000$ and $\mu_C = 6500$. You overhear a group of four graduates who had attended the same College and are now working for Zeta commenting that their average bonus is 7250.

(a) Perform a 5% significance test on the hypothesis that those graduates are from college B. Clearly state your null and alternative hypotheses, test statistic and its sampling distribution.

(b) What is the power associated with your test? (You may use the fact that the power is the same whether the graduates attended College A or C.)

Zeta's decision to promote each graduate is independent from one individual to another. At the end of each year any two graduates of the same College are equally likely to get promoted but this probability may differ between Colleges. Suppose 200 of 800 graduates from College A and 250 out of 750 graduates from College C got promoted.

(c) Perform a 5% significance test on the hypothesis that graduates from College A and C are equally likely to get promoted. Clearly state your null and alternative hypotheses, test statistic and its sampling distribution.

12 The total cost of producing good A, $C_i^A$, in plant $i$ depends on the quantity produced, $Q_i^A$, according to the following relationship:

$$C_i^A = \alpha^A + \beta^A Q_i^A + \varepsilon_i^A,$$

where $\{\varepsilon_i^A\}_{i=1}^n$ denotes a sequence of random draws from a normal distribution with mean zero and variance $\sigma_A^2$. The quantity produced in each plant can be treated as non-random, but differs across plants. Using observations of costs and quantities from 42 plants, the following linear regression is fitted using ordinary least squares

$$\widehat{C}^A = \underset{(10.2)}{24.3} + \underset{(1.06)}{3.11} Q^A, \quad \text{and } R_A^2 = 0.17$$

where () denotes the standard error of the parameter estimate immediately above it, and $R_A^2$ is the coefficient of determination.

(a) Interpret and comment on the estimates of $\alpha^A$ and $\beta^A$.

(b) What is the predicted level of *average* cost of producing good A when the quantity produced is 50?

(c) Perform a 5% significance test of the null hypothesis that the *marginal* cost of producing good A is 2 against the alternative hypothesis that it is larger than 2. Clearly state your null and alternative hypotheses, test statistic and its sampling distribution.

Suppose you also have 62 observations of costs and quantities from plants producing good B, where the total production cost, $C_i^B$, depends on the quantity produced, $Q_i^B$, according to the following relation:

$$C_i^B = \alpha^B + \beta^B \left(Q_i^B\right)^2 + \varepsilon_i^B,$$

where $\left\{\varepsilon_i^B\right\}_{i=1}^n$ denotes a sequence of random draws from a normal distribution with mean zero and variance $\sigma_B^2$. The quantity produced in each plant can be treated as non-random but may differ across plants. The following linear regression is fitted using ordinary least squares

$$\begin{aligned} \widehat{C}^B &= \underset{(5.1)}{20.4} + \underset{(2.03)}{4.72} \left(Q^B\right)^2, \quad \text{and } R_B^2 = 0.67 \end{aligned}$$

where () denotes the standard error of the parameter estimate immediately above it, and $R_B^2$ is the coefficient of determination.

(d) What is the predicted level of *marginal* cost for producing good B when the quantity produced is 50?

(e) Perform a 5% significance test of the null hypothesis that the marginal cost for producing good B does not change with quantity produced. Clearly state your null and alternative hypotheses, test statistic and its sampling distribution.

(f) Suppose that the random variables $\varepsilon_i^A$ and $\varepsilon_i^B$ associated with these two cost relationships are independent. Briefly discuss how you could test the hypothesis that the fixed costs of production are the same for both goods. (You are not required to attempt to carry out the test).

**END OF PAPER**