UNIVERSITY OF CAMBRIDGE

ECT1
ECONOMICS TRIPOS PART I

Friday 16 June 2017 9:00-12:00

Paper 3

QUANTITATIVE METHODS IN ECONOMICS

Answer **ALL FOUR** questions from Section A, **ONE** question from Section B, **ALL FOUR** questions from Section C, and **ONE** question from Section D.

Answers from Sections A and B (Mathematics) must be written in one booklet; answers from Sections C and D (Statistics) must be written in a separate booklet. Write the letters of the sections on each cover sheet.

Sections A and C each carry 30% of the total marks for this paper. Sections B and D each carry 20% of the total marks.

Each question within each section will carry equal weight.

Write your **candidate number** (not your name) on the cover of each booklet.

Candidates are asked to note that there may be a reduction in marks for scripts with illegible handwriting.

If you identify an error in this paper, please alert the **Invigilator**, who will notify the **Examiner**. A **general** announcement will be made if the error is validated.

STATIONERY REQUIREMENTS
20 Page booklet x 2
Rough work pads
Tags

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAMINATION
Calculator - students are permitted to bring an approved calculator
New Cambridge Elementary Statistical Tables

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A – Answer **ALL FOUR** questions from this Section.

1. (a) Find the Taylor series expansion for

$$f(x) = x^5 + x - 2$$

centred on $x = 1$.

(b) Let Y_t be national product at time t, C_t be total expenditure and I investment. Suppose $C_t = \alpha + \beta Y_t$, $\dot{Y}_t = \gamma(C_t + I - Y_t)$, I is a constant, and α, β and γ are positive constants.

i. Derive a differential equation for Y_t.

ii. Determine the value of any stationary point.

iii. Find a solution for Y_t if $Y_2 = 3$.

iv. Determine and interpret the conditions under which the system is stable.

2. Let $g(x) = (x-3)(x-2)(x-1)$

(a) Determine the values of x for which $g(x) > 0$, and also those for which $g(x) < 0$.

(b) Expand $g(x)$ expressing it in terms of x^3 and x^2 etc.

(c) Find the second order partial differentials of the function

$$f(x,y) = \frac{1}{20}\lambda x^5 - \frac{1}{2}\lambda x^4 + \frac{11}{6}\lambda x^3 - 3\lambda x^2 + 117x + \frac{\alpha}{\lambda}y^2 + 2456$$

(d) Using the results of (a) and (b) (or otherwise) determine values of x, α and λ for which the function $f(x,y)$ given in (c) is convex.

3. (a) "Matrix multiplication is not commutative in general, although it is associative; matrix addition is distributive."

Explain the above sentence. Is it correct?

(b) A and B are square matrices with $AB + BA = A$

Show that:

i. $A^2B - BA^2 = 0$

ii. $A^3B + BA^3 = A^3$

(c) Suppose

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 0 & 6 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 29 \\ 46 \\ 12 \end{pmatrix} \quad \text{and } A\mathbf{x} = \mathbf{y}$$

Using Cramer's rule (or otherwise), determine x_2.

4. Evaluate the following integrals:

(a)

$$\int_0^1 \left(\frac{e^{3x}}{(1+e^x)^2} + \frac{2e^x}{(1+e^x)} \right) dx$$

(b)

$$\int_1^5 (x-1)(5-x)^{\frac{1}{2}} dx$$

(TURN OVER)

SECTION B – Answer **ONE** question from this Section.

5. A monopolist can set prices differently in two isolated markets. The demand curves for each market and their total costs are given as:

$$\begin{array}{ll} \text{Market 1} & q_1 = 42.4 - 0.4p_1 \\ \text{Market 2} & q_2 = 20.6 - 0.1p_2 \end{array}$$

and

$$TC = 300 + 6\,(q_1 + q_2)$$

where q_i, and p_i are respectively quantity demanded and price of the commodity provided for market $i = 1, 2$. TC is total cost.

(a) Determine the maximum profit that can be achieved with price discrimination.

(b) Determine the price elasticity of demand in both markets at the profit maximising level of output.

(c) Determine the maximum profit that can be achieved without price discrimination.

(d) Compare and interpret your findings in (a) and (c).

(e) If a temporary quota on production is introduced at $Q = q_1 + q_2 = 8$ determine the maximised profits.

(f) Determine the price elasticity of demand at the point of maximal profits determined in (e).

6. An individual's utility from consuming x_1 units of good number 1 and x_2 units of good number 2 is:

$$U(x_1, x_2) = \alpha \ln(x_1 - a) + \beta \ln(x_2 - b)$$

where a, b, α and β are positive constants, $0 < \alpha,\ \beta < 1$.

(a) Determine the marginal rate of substitution of x_1 for x_2.

(b) If the utility level reached is constrained by the budget equation $px_1 + qx_2 = M$, where p and q are prices of units of goods 1 and 2 respectively, and if $pa + qb < M$, determine the demand curve for each good.

(c) How does an optimised value of utility change if M increases slightly. Simplify your answer as much as possible.

(d) Show that if $\alpha + \beta = 1$ then the demand curves can be expressed as:

$$\begin{aligned} px_1 &= \alpha M + pa - \alpha(pa + qb) \\ qx_2 &= \beta M + qb - \beta(pa + qb) \end{aligned}$$

(e) For either demand curve derived in (d), verify that the price elasticity of demand is negative.

SECTION C – Answer **ALL FOUR** questions from this Section.

7. X and Y are discrete random variables whose joint probability mass function is given (partially) in the table below. The marginal distributions are such that $P(X=-1) = 1/3$ and $P(Y=-1) = P(Y=0) = 1/4$.

		Y		
		-1	0	$+1$
X	-1	$1/48$		
	$+1$			$1/4$

(a) Complete the table and explain whether X and Y are independent random variables.

(b) Write out the probability mass functions for the random variables $\frac{Y}{X}$ and $\frac{1}{X}$.

(c) Calculate $E(X)$, $E(Y)$, $E\left(\frac{Y}{X}\right)$ and $E\left(\frac{1}{X}\right)$.

(d) Hence or otherwise calculate $Cov\left(Y, \frac{1}{X}\right)$.

8. (a) You purchase a certain product. The manual states that the lifetime, T, of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = exp\left(-\frac{t}{5}\right), \text{ for all } t \geq 0$$

If you use it for two years without any problems, what is the probability that it breaks down in the following year (i.e. between $t=2$ and $t=3$)?

(b) According to the Centers for Disease Control (CDC), women who smoke are about 13 times more likely to develop lung cancer than women who do not smoke. They also report that in 2015 13.6% of women were smokers. If you learn that a woman has been diagnosed with lung cancer, and you know nothing else about her, what is the probability that she is a smoker?

9. A continuous random variable X has probability density function given by

$$f(x) = \frac{1}{\theta}, \quad \text{for } 0 \leq x \leq \theta$$

where $\theta > 0$ is some parameter.

(a) Show that this function satisfies the definition of a probability density function.

(b) Calculate $E(X)$ and $E(X^2)$ and hence $Var(X)$.

(c) What is the cumulative distribution function for the random variable X?

(d) Given an IID random sample $X_1, \ldots, X_n$ of observations from $f(x)$, calculate the expected value and variance of the sample average given by

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

(e) Give an unbiased estimator for θ. Explain why it is unbiased and calculate its variance.

10. Consider the linear population regression model $y = \beta_0 + \beta_1 x + u$.

(a) State the Gauss-Markov assumptions.

Let $z = log(1 + x^2)$. Define an estimator for β_1 as

$$\tilde{\beta}_1 = \frac{\sum_{i=1}^{N}(z_i - \overline{z})y_i}{\sum_{i=1}^{N}(z_i - \overline{z})x_i}$$

(b) Assume that the Gauss-Markov assumptions hold. Show that $\tilde{\beta}_1$ is an unbiased estimator for β_1.

(c) Next, assume that the assumption of constant conditional variance is violated. Explain how this affects your answer to part (b).

SECTION D – Answer **ONE** question from this Section.

11. Workers in a firm sell a product. The value of a worker's sales is a random variable X with exponential distribution. The exponential distribution has probability density function $f(x) = \frac{1}{\beta}e^{-x/\beta}$, where $x \geq 0$ and $\beta > 0$ is a parameter.

(a) For the exponential distribution calculate $E(X)$ and $E(X^2)$ and hence $Var(X)$.

(b) Initially the firm offers a bonus payment to any worker whose sales exceed 2. What fraction of the workforce (as a function of β) will get the bonus?

(c) The firm observes average sales of a sample of 100 workers to be $\bar{x}$. Stating clearly any assumptions you make, what can you say about the distribution of $\bar{x}$?

(d) For a test of a hypothesis H explain what is meant by

i. a Type I and Type II error,

ii. the size and power of the test.

(e) The firm thinks that its workers either all have β equal to $1/3$ or all have β equal to $1/2$. Test, at the 1% level, the null hypothesis that $\beta = 1/3$ against the alternative $\beta = 1/2$ if the realised value of $\bar{x} = 0.433$.

(f) What is the power of this test against the stated alternative?

(g) The firm then decides it should only pay a bonus to 5% of the workers. If half the workforce is of each type, what threshold should be chosen? You may assume without proof that the cubic equation $x^3 - 10x - 10 = 0$ has a single positive solution at $x = 3.578$.

12. Consider the regression $lwage_i = \alpha + \beta_1 educ_i + \varepsilon_i$ where *lwage* is the natural log of the wage, and *educ* is years of education. An economist estimates this regression using a nationally representative sample from the population of a high income country.

Table 1

```
. reg lwage educ

      Source |       SS       df       MS              Number of obs =     864
-------------+------------------------------           F(  1,   862) =   91.61
       Model |  22.2629147     1  22.2629147           Prob > F      =  0.0000
    Residual |   WWWW        862  .243022541           R-squared     =   TTTT
-------------+------------------------------           Adj R-squared =   SSSS
       Total |  231.748345   863  .268538059           Root MSE      =  .49297

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        educ |    .055974   .0058481      VVVV    XXXX       YYYY         ZZZZ
       _cons |   1.195559   .0771467     15.50   0.000     1.044141    1.346976
------------------------------------------------------------------------------
```

(a) What is the estimate of β_1 in this sample? How do you interpret the estimate?

(b) Calculate VVVV, YYYY, and ZZZZ in the Table above.

(c) Given your answer to part (b), what is a likely value for XXXX? Interpret the p-value associated with the estimate of the coefficient on *educ*.

(d) Calculate WWWW. Explain what Residual SS is.

(e) Provide a mathematical formula for R^2. Calculate its value for this regression.

Next the same economist uses Stata to generate the output in Table 2 below.

Table 2

```
. reg lwage

      Source |       SS       df       MS              Number of obs =     864
-------------+------------------------------           F(  0,   863) =    0.00
       Model |           0     0           .           Prob > F      =       .
    Residual |  231.748345   863  .268538059           R-squared     =  0.0000
-------------+------------------------------           Adj R-squared =  0.0000
       Total |  231.748345   863  .268538059           Root MSE      =  .51821

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |   1.916288   .0176297   108.70   0.000     1.881686    1.950891
------------------------------------------------------------------------------

. predict lwage_pred
(option xb assumed; fitted values)

. gen lwage_dev=lwage-lwage_pred
```

(f) Write out the regression function for the Stata output from Table 2. Interpret the constant term.

(g) Using the information in Tables 1 and 2, roughly sketch a scatterplot of the residuals from the regression in Table 2 in which the residuals are on the y-axis and year of education is on the x-axis. Explain any pattern one would observe.

The same economist then uses Stata again to generate the output in Table 3 below.

 (TURN OVER)

Table 3

```
. reg educ

      Source |       SS       df       MS              Number of obs =     864
-------------+------------------------------           F(  0,   863) =    0.00
       Model |           0     0           .           Prob > F      =       .
    Residual |  7105.74884   863  8.23377618           R-squared     =  0.0000
-------------+------------------------------           Adj R-squared =  0.0000
       Total |  7105.74884   863  8.23377618           Root MSE      =  2.8695

------------------------------------------------------------------------------
        educ |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _cons |   12.87616   .0976209   131.90   0.000     12.68456    13.06776
------------------------------------------------------------------------------

. predict educ_pred
(option xb assumed; fitted values)

. gen educ_dev=educ-educ_pred
```

Finally, the economist estimates

```
. reg lwage_dev educ_dev
```

(h) Explain what the estimated coefficient on *educ_dev* will be. Use the definition of the slope parameter in a bivariate regression to provide a derivation that justifies your answer.

(i) How does the value of R^2 in this last regression compare to that of the R^2 you calculated in part (e) above for the regression of *lwage* on *educ*? Explain.

END OF PAPER