PART IIA EXAMINATION OF THE ECONOMICS TRIPOS

Thursday 5 June 2003 9 to 12

Paper 6

MATHEMATICS

The paper consist of two Sections; A and B.

Each Section carries 50% of the total marks

Candidates may attempt **SIX** *questions from Section A, and* **THREE** *questions from Section B.*

Credit will be given for complete answers; answers to individual parts of questions will gain less than pro rata credit.

Write on **one** *side of the paper only.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. Consider the system of equations

$$\begin{aligned} -3x + 4y &= 8 \\ 6x + ty &= s \end{aligned}$$

where t and s are real numbers.

   (a) Write the matrix equation for this system

   (b) Find values for t and s so that the system has one and only one solution

   (c) Find values for t and s so that the system has no solutions

   (d) Find values for t and s so that the system has infinitely many solutions

   (e) Give a geometric interpretation of (b), (c) and (d).

2. Answer both parts:

   (a) Compute the missing entries:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ * & -1 & -1 \\ * & * & -1 \end{bmatrix}$$

   (b) Use the result above to compute the inverse of the following matrix

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & -9 \\ 0 & 1 & -1 \end{bmatrix}$$

3. A square matrix $\mathbf{A}$ is orthogonal iff $\mathbf{A}^{-1} = \mathbf{A}^{\top}$. Answer both parts:

   (a) Let $\mathbf{A}$ and $\mathbf{B}$ be $(n \times n)$ orthogonal matrices. Show that $\mathbf{AB}$ is an orthogonal matrix.

   (b) Show that if $\mathbf{A}$ is an orthogonal matrix, then $\det(\mathbf{A})$ is either 1 or -1.

4. (a) Find and classify the critical points and sketch the following function:

$$f(x) = x + \frac{1}{x},\ x \neq 0$$

Which critical point is greater? Comment.

(b) Find and classify the critical points of the following function

$$f(x, y) = 3x^2 - 6xy + y^2 + y^4$$

Evaluate such function at the critical points.

5. Find the solution of the differential equation

$$\ddot{y} - 2\dot{y} + (1 - h^2)y = 0,\ h \neq 0$$

that satisfies the initial condition $y(0) = 0$ and $\dot{y}(0) = 1$. Use l'Hôpital's rule to find the solution when $h = 0$.

6. Let $f(\mathbf{x})$ be homogeneous of degree n and let $g(\mathbf{x})$ be homogeneous of degree k. Let $\mathbf{x}^*$ be the stationary points of f, subject to the constraint $g(\mathbf{x}) = c$. Show that

$$f(\mathbf{x}^*) = \frac{\lambda k c}{n}$$

where λ is the Lagrange multiplier.

7. A series of independent experiments each characterised by two outcomes, *success* and *failure* has Pr(*success*) $= p$ and Pr(*failure*) $= 1 - p$.

   (a) Find an expression for $\Pr(X = x)$, $x = 1, 2, 3, ...$, where X is a random variable denoting the number of experiments until the first success occurs. Verify that this expression represents a valid probability function.

   (b) By finding an expression for $\Pr(X > t)$ demonstrate the memoryless property of the geometric distribution. Namely, show that $\Pr(X > t + k | X > t) = \Pr(X > k)$.

(TURN OVER

8. Independent normal random variables X and Y have means μ and $\rho\mu$ respectively, and variance 1. The ratio ρ is of particular interest.

   (a) Find the distribution of $Y - \rho X$.

   (b) A single observation is made on X and Y. Using your findings from (a) test the null hypothesis $\rho = 1$ against an alternative $\rho \neq 1$ at the 5% level when:
   (i) $x = 0.3$, $y = 2.7$;
   (ii) $x = 0.3$, $y = 1.5$.

9. An analyst observes a random sample, $\mathbf{z} = (z_1, z_2, ..., z_n)$, of a random variable Z, with probability density function depending upon an unknown mean parameter θ.

   (a) For n large, use the pivotal quantity method to find an expression for the classical $100(1-\alpha)\%$ interval estimator for θ. Provide a brief explanation of the information conveyed by such an estimator.

   (b) Another analyst suggests that a preferable interval estimator can be derived from the prior probability and conditional probability distribution of θ - namely $g(\theta)$ and $f(\theta|\mathbf{z})$. In what sense is this estimator distinct from the classical interval estimator in (a) ?

SECTION B

1. A town is served by two newspapers, the Star and the Mail. Each year the Star loses 40% of its subscribers to the Mail and retains 60% of its subscribers. During the same period, the Mail loses 10% of its subscribers to the Star while retaining the other 90%.

   (a) Write down the transition matrix that describes the transition of subscribers between the two papers each year.

   (b) How would you compute the number of subscribers to each paper in any given year (as a function of time only)?

   (c) In the long run, approximately what percentage of the subscribers will subscribe to the Mail?

2. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $(n \times n)$ matrices with the properties that $\mathbf{AB} = \mathbf{BA}$ and $Nul\,(\mathbf{A}) = Nul\,(\mathbf{B})$ (i.e. their null spaces are the same).

   (a) Show that if $\mathbf{v}$ is an eigenvector for $\mathbf{A}$ corresponding to the non-zero eigenvalue λ, then $\mathbf{Bv}$ is also an eigenvector for $\mathbf{A}$ corresponding to the same eigenvalue λ.

   (b) Suppose that $\mathbf{A}$ has distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$. Show that $\mathbf{B}$ is diagonalizable. (*Hint: Show that* $\mathbf{A}$ *and* $\mathbf{B}$ *have the same eigenvectors*).

(TURN OVER

3. Determine the equilibrium points and analyze their local stability for the following non-linear systems:

   (a)
$$x' = y^2 - 3x + 2$$
$$y' = x^2 - y^2$$

   (b)
$$x' = 1 - e^y$$
$$y' = 3x - y$$

   (c)
$$x' = x^3 + 3x^2y + y$$
$$y' = x(2 + y^2)$$

4. The output of a process is given by $Q = F(K, L)$ where K and L are inputs. Write an expression for the change in output dQ in terms of dK and dL and the marginal products.

   (a) Given that $F(K, L) = K^{1/3}L^{2/3}$ and $dK/K = dL/L = c$, find in terms of c the approximate fractional change in output.

   (b) For
$$F(K, L) = [\alpha K^m + \beta L^m]^n$$
   show that this function is homogeneous and state the degree of homogeneity. State Euler's Theorem and verify that it holds for this function.

   (c) A monopoly manufactures two goods of which it sells quantities x_1 and x_2 at prices p_1 and p_2 respectively. The demand functions are $D_1(\mathbf{p}) = p_2 - p_1$ and $D_2(\mathbf{p}) = 9 + p_2 - 2p_1$, where we set $\mathbf{p} = (p_1, p_2)'$. The manufacturing unit costs of the goods are 2 and 3 respectively. Show that the profit function is
$$\pi(x_1, x_2) = x_1^2 - x_2^2 + x_1x_2 + 7x_1 + 6x_2.$$
   Maximize π and confirm that your solution is a maximum.

5. A continuous random variable X has an exponential distribution with a probability density function

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

The mean and variance of this distribution are λ^{-1} and λ^{-2} respectively.

(a) Find, in terms of λ, numbers a and b such that $\Pr(X < a) = \Pr(X > b) = 0.05$, and therefore determine $\Pr(a \leq X \leq b) = 0.90$.

(b) From (a) determine a 90% confidence interval for λ^{-1} based upon a single observation of the random variable X.

(c) Let $\bar{X}$ denote the mean of a random sample of 81 observations on X. State the approximate distribution of $\bar{X}$, and show that

$$\Pr(0.821\bar{X} \leq \lambda^{-1} \leq 1.278\bar{X}) = 0.95$$

6. A discrete random variable Y takes the values -1, 0, and 1 with probabilities $\frac{1}{2}\theta$, $1-\theta$, and $\frac{1}{2}\theta$ respectively. Let Y_1 and Y_2 be two independent random variables, each with the same distribution as Y.

(a) By listing the set of possible values of $\{Y_1,Y_2\}$ find the joint probability distribution function $\Pr(Y_1 = y_1, Y_2 = y_2)$. Verify that this is a valid probability distribution function.

(b) Find the sampling distribution of $(Y_2 - Y_1)^2$.

(c) Show that $X = \frac{1}{2}(Y_2 - Y_1)^2$ is an unbiased estimator for θ.

END OF PAPER