PART IIA EXAMINATION OF THE ECONOMICS TRIPOS

Thursday 3 June 2004 9-12

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

The paper consist of two Sections; A and B.

Each Section carries 50% of the total marks

Candidates may attempt **SIX** *questions from Section A, and* **THREE** *questions from Section B.*

Credit will be given for complete answers; answers to individual parts of questions will gain less than pro rata credit.

Write on **one** *side of the paper only.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. For each output level Y, the IS curve defines the interest rate r at which the goods market clears:

$$Y(1-b) - G = I^0 - ar,$$

where b is the marginal propensity to consume; G is the government spending, I^0 is the maximum investment level; and a is the responsiveness of investment to interest rates. The LM curve defines the interest rate at which the money market clears:

$$mY + M^0 - hr = M^s,$$

where m is the responsiveness of the transaction demand for money to output, M^0 is the maximum liquidity demand, h is the responsiveness of the liquidity demand to interest rates, and M^s is the money supply.

   (a) Write down this system of equations in matrix form. Under what condition on the exogenous parameters can this system of two equations be solved for Y and r?

   (b) Using Cramer's rule, solve the system for Y and r when the condition in (a) is met.

   (c) What happens to the equilibrium interest rate r if government spending increases by ΔG?

2. Consider the 4×4 matrix

$$A = \begin{bmatrix} c+1 & 2 & 0 & 3 \\ 0 & c & 0 & 6 \\ 7 & 8 & 2 & c \\ 0 & 0 & 0 & 12 \end{bmatrix},$$

where c is a constant.

   (a) Write $\det(A)$ in terms of c.

   (b) Assuming that the inverse matrix A^{-1} exists, compute the element on the fourth row and the third column of A^{-1}.

3. Consider three vectors

$$v_1 = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} c \\ -1 \\ 5 \end{bmatrix},$$

where c is a constant.

(a) For what values of c is it true that $v_3 \in \text{Span}\,[v_1, v_2]$?

(b) What is the dimension of $\text{Span}\,[v_1, v_2, v_3]$?

(c) Is there any value for c such that both 0 and 3 are eigenvalues of the 3×3 matrix $[v_1\ v_2\ v_3]$?

4. For two functions $f(x) = x^3 - cx^2 - x$ and $g(x) = x - 1$, where c is a constant, does the fraction

$$\frac{f(x)}{g(x)}$$

converge to any number as $x \to 1$, and, if so, which one?

5. Let $F(x, y)$ be a function of two variables and c be a constant such that the absolute values of all second-order partial derivatives,

$$\left|\frac{\partial^2 F(x,y)}{\partial x^2}\right|, \left|\frac{\partial^2 F(x,y)}{\partial x \partial y}\right|, \left|\frac{\partial^2 F(x,y)}{\partial y \partial x}\right|, \left|\frac{\partial^2 F(x,y)}{\partial y^2}\right|$$

cannot exceed c at any (x, y). Find an error bound (that is, an upper bound on the Lagrange remainder) of the approximation by the Taylor series around (x, y) up to the first order for $F(x + h, y + k)$, where $|h| \leq 1/3$ and $|k| \leq 1/3$.

6. Solve the following coupled first-order linear differential equations

$$\begin{aligned} \dot{x}(t) &= x(t) + y(t), \\ \dot{y}(t) &= cy(t), \end{aligned}$$

with the initial conditions $x(0) = 0$ and $y(0) = 1$, where c is a constant.

7. In this question, we say that event A *attracts* event B if $\Pr(B|A) > \Pr(B)$ and A *repels* B if $\Pr(B|A) < \Pr(B)$

   (a) Prove that A attracts B if and only if B attracts A.

   (b) Prove that A attracts B if and only if $\Pr(B|A) > \Pr(B|\tilde{A})$, where $\tilde{A}$ is the complement of A.

   (c) Prove that if A attracts both events B and C, and A repels $B \cap C$, then A attracts $B \cup C$.

8. Let X be a random variable with mean μ and standard deviation σ and define the standardized version X^* of X by $X^* = (X - \mu)/\sigma$.

   (a) Prove that if $X \geq 0$, then $\Pr(X \geq c) \leq E[X]/c$ for every $c > 0$.

   (b) Prove that $\Pr(|X^*| \geq c) \leq 1/c^2$ for every $c > 0$.

   (c) Let $X_1, \ldots, X_n$ be a random sample of n observations on the random variable X. Prove that $(X_1 + \cdots + X_n)/n$ converges in probability to μ as $n \to \infty$.

9. Suppose that you are using Bayesian techniques to test two hypotheses, H_1 and H_2.

   (a) Show that the posterior odds of H_1 relative to H_2 may be written

$$\frac{\Pr(H_1|\mathbf{x})}{\Pr(H_2|\mathbf{x})} = \frac{\Pr(H_1)}{\Pr(H_2)} \frac{L(H_1|\mathbf{x})}{L(H_2|\mathbf{x})},$$

   where $\Pr(H_1)/\Pr(H_2)$ is the ratio of prior probabilities, $L(H_1|\mathbf{x})/L(H_2|\mathbf{x})$ is the likelihood ratio, and $\mathbf{x}$ denotes the data.

   (b) Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample from a Bernoulli distribution with success rate θ, of which the prior distribution is given by the probability density function

$$p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \quad (0 \leq \theta \leq 1).$$

   Find its posterior distribution and give the name of the posterior distribution.

   (c) Using the fact that, for $m \leq n$, the marginal distribution of $(x_1, \ldots, x_m)$ is given by the probability mass function

$$p(x_1, \ldots, x_m) = \int_0^1 \prod_{i=1}^{m} \theta^{x_i}(1-\theta)^{1-x_i} p(\theta)\, d\theta,$$

   show that the conditional distribution of $(x_{m+1}, \ldots, x_n)$ conditional on $(x_1, \ldots, x_m)$ is given by the probability mass function

$$p(x_{m+1}, \ldots, x_n \mid x_1, \ldots, x_m) = \int_0^1 \prod_{i=m+1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} p(\theta|x_1, \ldots, x_m)\, d\theta,$$

   where $p(\theta|x_1, \ldots, x_m)$ is the posterior distribution of θ conditional on $(x_1, \ldots, x_m)$.

SECTION B

1. Let A be an $m \times n$ matrix and define a linear mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ by $F(x) = Ax$. The kernel of F, $\mathrm{Ker}\,(F)$, is defined as the linear subspace $\{x \in \mathbb{R}^n : F(x) = 0\}$ and the image of F, $\mathrm{Im}\,(F)$, is defined as the linear subspace $\{y \in \mathbb{R}^m :$ there exists an $x \in \mathbb{R}^n$ such that $F(x) = y\}$.

   (a) Using the fundamental theorem of linear algebra (which provides a relationship between the dimension of the null space and the rank of a matrix), prove that

   $$n = \dim(\mathrm{Ker}\,(F)) + \dim(\mathrm{Im}\,(F))\,.$$

   (b) Define another linear mapping $F^{\mathrm{T}} : \mathbb{R}^m \to \mathbb{R}^n$ by $F^{\mathrm{T}}(y) = A^{\mathrm{T}}y$. Prove that for every $y \in \mathbb{R}^m$, we have $y \in \mathrm{Ker}\,(F^{\mathrm{T}})$ if and only if

   $$y \cdot z = y_1 z_1 + \cdots + y_m z_m = 0$$

   for every $z \in \mathrm{Im}\,(F)$.

   (c) Now let

   $$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 6 & 10 & -4 \\ 3 & 8 & 13 & -3 \end{bmatrix},$$

   and find $\dim(\mathrm{Im}\,(F))$ and a basis of $\mathrm{Ker}\,(F)$.

2. Consider a stock whose price process is Markov. The price on each period is either £15 or £30. If the price is £15 in the current period, then the probability that it will remain at the same level in the next period is 0.8. If the price is £30 in the current period, then the probability that it will remain at the same level in the next period is 0.6.

   (a) Write down the transition matrix for this Markov process.

   (b) Find the eigenvalues and eigenvectors of the transition matrix.

   (c) What is the average stock price in the long run?

3. Consider a function of two variables,

$$F(x, y) = ax^{2n} - 2bx^n y^n + cy^{2n},$$

where a, b, and c are constants, and n is a positive integer.

(a) Prove that $(0, 0)$ is a stationary point.

(b) If $a = b = c \neq 0$, find all stationary points, depending on whether n is odd or even.

(c) If $n = 1$, for what values of a, b, and c is the Hessian matrix at $(0, 0)$ negative semi-definite?

(d) If $b = 0$, for what values of a, c, and n is the Hessian matrix at $(0, 0)$ positive definite?

(e) If $a = c$, for what values of a and b is $(0, 0)$ a saddle point for $n = 1$ and, at the same time, a global minimum point for $n = 2$?

4. Let $U(x_1, x_2)$ be a concave utility function over two goods and $h(z)$ be a strictly increasing function of one variable. Define two other utility functions $F(x_1, x_2)$ and $G(x_1, x_2)$ by

$$\begin{aligned} F(x_1, x_2) &= h(U(x_1, x_2)), \\ G(x_1, x_2) &= U(h(x_1), h(x_2)). \end{aligned}$$

(a) Is $F(x_1, x_2)$ concave or quasi-concave? If so, prove it. If not, give a counter-example.

(b) Is $G(x_1, x_2)$ concave or quasi-concave? If so, prove it. If not, give a counter-example.

(c) If $U(x_1, x_2)$ and $h(z)$ are homogeneous (with possibly different degrees), is $G(x_1, x_2)$ homogeneous? If so, what is its degree?

(d) Denote by $V(p_1, p_2, m)$ the indirect utility function corresponding to $U(x_1, x_2)$ and by $W(p_1, p_2, m)$ the indirect utility function corresponding to $F(x_1, x_2)$, where m denotes the income level. Prove that

$$\frac{\partial W(p_1, p_2, m)}{\partial m} = h'(V(p_1, p_2, m)) \frac{\partial V(p_1, p_2, m)}{\partial m}.$$

5. Suppose that a random variable X follows the Poisson distribution with parameter $\theta > 0$. Its probability mass function is therefore

$$f(x) = e^{-\theta}\frac{\theta^x}{x!}$$

for $x = 0, 1, 2, \ldots$.

   (a) Show that the first moment about the origin and the second moment about the mean are both equal to θ.

   (b) Find the distribution of a random sample of n observations from the Poisson distribution.

   (c) Find the log-likelihood function of a random sample of n observations and show that the maximum likelihood estimator for θ is unbiased.

   (d) Using the Cramer-Rao lower bound, determine whether the maximum likelihood estimator is fully efficient.

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution with an unknown parameter θ.

   (a) Show that the mean square error of any estimator is equal to the sum of its variance and squared bias.

   (b) When the sample is from the normal distribution with mean θ and variance 1, find the mean square error of the estimator $(1/n)(X_1 + \cdots + X_n)$ for θ and show that it is consistent.

   (c) When the sample is from the uniform distribution over the interval $[0, \theta]$, compare the mean square errors of two estimators $(1/n)(X_1 + \cdots + X_n)$ and $(2/n)(X_1 + \cdots + X_n)$ for θ and determine which one, if either, is consistent.

   (d) When the sample is from the binomial distribution with success rate θ, we define, for each positive integer k, the estimator $\widehat{\theta}_k$ as $(1/k)(X_1 + \cdots + X_n)$. Compare the mean square errors of $\widehat{\theta}_{n-1}$, $\widehat{\theta}_n$, and $\widehat{\theta}_{n+1}$, and comment on your result.

END OF PAPER