PART IIA EXAMINATION OF THE ECONOMICS TRIPOS

Thursday 9 June 2005 9-12

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections; A and B.

Each Section carries 50% of the total marks.

Candidates may attempt **SIX** *questions from Section A, and* **THREE** *questions from Section B.*

Credit will be given for complete answers; answers to individual parts of questions will gain less than pro rata credit.

Write on **one** *side of the paper only.*

STATIONERY REQUIREMENTS
20 Page Booklet
Rough Work Pads

SPECIAL REQUIREMENT
Approved calculators allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. Consider the following three vectors in $\mathbb{R}^3$

$$v_1(c) = \begin{bmatrix} 1 \\ c \\ -4 \end{bmatrix}, \; v_2(c) = \begin{bmatrix} 1 \\ 6 \\ c \end{bmatrix}, \; v_3(c) = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix},$$

where $c \in \mathbb{R}$.

(a) For what values of c are the three vectors linearly dependent? (You should find two values c_1, c_2)

(b) For c_1 and c_2 calculated in (a) find a basis for $V(c_i) = span\,\langle v_1(c_i), v_2(c_i), v_3(c_i)\rangle$, $i = 1, 2$.

2. Consider the following 4×4 matrix:

$$A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 0 & 5 & 0 & 3 \end{bmatrix}$$

(a) If

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

with $x \in \mathbb{R}^4$, what is x_2?

(b) What is the dimension of $Col(A)$? What is the dimension of $Null(A)$?

3. Consider a $n \times n$ matrix A. A is *idempotent* if

$$AA = A$$

Show that if A is idempotent, then

(a)

$$\det A \neq 0 \text{ if and only if } A = I,$$

where I is the $n \times n$ identity matrix.

(b) If A can be written as

$$A = PDP^{-1},$$

where D is a $n \times n$ diagonal matrix, then all eigenvalues of A must be either 0 or 1.

4. Let $f(K, L)$ be a production function with constant returns to scale, where K denotes capital, and L denotes labour.

   (a) Show that if we scale both input factors up or down by $t > 0$, the marginal products of labour and capital remain the same.

   (b) Show that
   $$f_{11}(K, L) K + f_{12}(K, L) L = 0,$$
   for all K and L.

5.

   (a) By considering the Hessian matrix show that $x^a y^b$ is concave if $a+b < 1$, where $x > 0, y > 0, a > 0, b > 0$.

   (b) Solve the following constrained optimisation problem:
   $$\begin{array}{c} \max_{x,\ y} -y \\ s.t.\ y^3 - x^2 = 0. \end{array}$$

6. Find the specific solution for the following differential equations. In each case, explain what happens to the dependent variables as $t \longrightarrow \infty$.

   (a)
   $$\dot{y} = \frac{y}{7t^2};\ y(1) = 1.$$

   (b)
   $$\ddot{y} + \dot{y} = 0;\ y(0) = 2,\ y'(0) = 3.$$

   (c)
   $$\dot{x} = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix} x;\ x(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}.$$

7. A Pollster wishes to obtain information on intended voting behavior in a two party system, and samples a *fixed* number (n) of voters. Let X_1,...,X_n denote the sequence of independent Bernoulli random variables representing voting intention, where $E(X_i) = p$, $i = 1, .., n$, and $X = \sum_{i=1}^{n} X_i$.

   (a) Find $E(X)$ and $Var(X)$.

   (b) Instead of fixing the number of individuals to interview, the pollster now samples N individuals, where N is a discrete random variable with probability mass function $p(n) = P(N = n)$.

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i. Find an expression for the conditional distribution of X given $N = n$.

ii. Find $E(X \mid N)$ and $Var(E(X \mid N))$.

(c) Using the result that for random variables, X and N

$$Var(X) = Var(E(X \mid N)) + E(Var(X \mid N)),$$

show that $Var(X) = p^2 Var(N) + p(1-p)E(N)$.

8. Suppose that X_1 and X_2 are two random variables with respective means μ_1 and μ_2, and variances, σ_1^2 and σ_2^2. Define the random variables $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

   (a) Compute the mean and variance of Y_1 assuming that X_1 and X_2 are *independent.*

   (b) Show that if X_1 and X_2 are *identically distributed*, then Y_1 and Y_2 are uncorrelated.

   (c) Consider now the case where X_1 and X_2 are independent and identically distributed Bernoulli variables with probability $\frac{1}{2}$. Can we conclude that Y_1 and Y_2 are independent?

9. Each month an economist chooses between one of three models in making a prediction. She uses model M_1 with probability 0.5, M_2 0.25, and M_3 0.25. Using M_1 there is an 80% chance that the prediction is accurate; corresponding figures for M_2 and M_3 are 40% and 60% respectively.

   (a) Find the probability that, in a given month, the prediction is accurate.

   (b) Find the probability that the forecast is accurate on at least 2 of 3 consecutive months.

   (c) If in a given month the prediction has not been accurate, what model has she most likely used?

   (d) Another economist also makes a forecast every month, and chooses between the three models with equal probabilities. Find the probability that the two economists will use the same model at least once in the next two months.

SECTION B

1. Consider a symmetric $n \times n$ matrix A: $A = A^T$. Assume that A has n distinct eigenvalues $r_1, ..., r_n$ with corresponding eigenvectors $v_1, ..., v_n$. Show that

   (a) Eigenvectors are mutually orthogonal, i.e.

   $$v_i \cdot v_j = 0, \ i \neq j.$$

   (b) There exists an orthogonal matrix P: $P^T P = I$, where I is the $n \times n$ identity matrix, such that

   $$P^T A P = \begin{bmatrix} r_1 & 0 & ... & 0 \\ 0 & r_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & r_n \end{bmatrix}.$$

   (c) For the following matrix A, compute the eigenvalues, eigenvectors and the matrix P.

   $$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

2. Consider two subspaces U_1 and U_2 of a vector space V. Define the *intersection* and the *sum* of U_1 and U_2 by

   $$\begin{aligned} U_1 \cap U_2 &= \{u \mid u \in U_1, \, u \in U_2\} \\ U_1 + U_2 &= \{u_1 + u_2 \mid u_1 \in U_1, \, u_2 \in U_2\}. \end{aligned}$$

   (a) Prove that $U_1 \cap U_2$ and $U_1 + U_2$ both are subspaces of V.

   (b) For the following matrix A, what is $Null(A) \cap Col(A)$ and what is $Null(A) + Row(A)$?

   $$A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

   (c) Explain why if $Ax = b$ has a solution it will have an affine subspace of solutions of dimension 1, where A is the matrix in part (b).

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3. Let $\Phi(x)$ be the cumulative distribution function of the standard normal distribution, i.e. $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$.

   (a) Use L' Hôpital's rule to show that
   $$\lim_{x \longrightarrow -\infty} x\Phi(x) = 0.$$

   (b) Show that $sign\left(\frac{d}{dx}\left(\frac{\Phi(x)}{\Phi'(x)}\right)\right) = sign\left(\Phi'(x) + \Phi(x)\,x\right)$, where $sign\left(f(x)\right)$ is the sign of $f(x)$.

   (c) Use (a) and (b) to show that $\frac{\Phi(x)}{\Phi'(x)}$ is increasing in x over $\mathbb{R}$.

4. Let x denote a consumption bundle, p the price vector, and m the consumer's income. Let $u(x)$ denote a consumer's utility function, and u denote a certain utility level. Let $V(p, m)$ be the indirect utility function, and $E(p, u)$ the expenditure function. Assume that $m = E(p, u)$.

   (a) Let $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$. Compute $V(p, m)$ and $E(p, u)$.

   (b) For general utility functions, let $D(p, m)$ denote the Mashallian demand function. Show that
   $$D(p, m) = -\frac{V_p(p, m)}{V_m(p, m)}.$$

   (c) For general utility functions, let λ and μ denote the Lagrangian multipliers of the utility maximisation and expenditure minimisation problems, respectively. Show that $\lambda\mu = 1$.

5. In seeking to estimate the proportion, P, of vehicles which run on non-fossil fuels, a student surveys 5000 motorists at random and counts X, the number of such cars. The students' estimate for P is $\widehat{P} = X/5000$.

   (a) Prove whether or not $\widehat{P}$ is a maximum likelihood estimator.

   (b) Prove whether or not $\widehat{P}$ is an unbiased estimator.

   (c) Use the central limit theorem to find a 95% confidence interval for P in terms of $\widehat{P}$.

6. Shirley runs a real estate company. She counts the total number of flats that she sells every day, X_i, and the total number of flats she sells in a week, X. Suppose that the X_i are i.i.d. random variables with a Poisson distribution, i.e.

$$\Pr(X_1 = x) = \frac{\theta^x e^{-\theta}}{x!}, x = 0, 1, ..., \qquad E(X_1) = \theta.$$

(a) Derive the moment generating function for X_1 and X.

(b) Show that X has a Poisson distribution with mean 7θ.

(c) Shirley's partners are afraid that the company cannot sell, on average one house every two days i.e. 0.5 house per day. In the past week sales were:

i	1	2	3	4	5	6	7
x_i	0	1	2	0	3	2	0

Compute $\Pr(X \geq 8|\theta = 0.5)$. Can Shirley refute her partners' worries based on this past week's data?

END OF PAPER