PARTIIA EXAMINATION OF THE ECONOMICS TRIPOS

Thursday 8 June 2006 9.00 - 12.00

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections; A and B.

Each Section carries 50% of the total marks.

Candidates should attempt ***SIX*** *questions from Section A and* ***THREE*** *questions from Section B*

Write on **one** *side of the paper only.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 2	Approved calculators allowed
Rough Work pads	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. Suppose X_n and Y_n are two estimators for θ, a real number. Suppose $\mathbb{E}X_n = \theta$, $Var\,(X_n) = cn^{-1}$, where c is a real positive constant, and $\mathbb{E}Y_n = \theta + n^{-1}$, $Var\,(Y_n) = n^{-1}$.

   (a) Compute the mean square error of X_n and of Y_n about θ.

   (b) Using the mean square error criterion, for what values of c will you choose X_n over Y_n?

2. Suppose $X_1, ..., X_n$ is a random sample from a distribution with finite second moment and mean μ. Give the definition of an unbiased estimator. Give the definition of a consistent estimator. Prove that $\bar{X}_n := n^{-1}\sum_{i=1}^{n} X_i$ is consistent for μ.

3. (a) For any two random variables X and Y show that

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

   (b) Consider performing independent Bernoulli trials $X_1, X_2, ...$, where the number of trials, N, is not fixed but is a discrete random variable, with probability mass function $p(n) = P(N = n)$. Find $E(X|N)$, $Var(E(X|N)$,and $Var(X)$ for X distributed Bernouilli

4. (a) Suppose we have

$$\lim_{x\longrightarrow\infty}\left(\frac{x^2+1}{x+1} - ax - b\right) = 0,$$

   what should a and b be?

   (b) Show that the equation $x^3 - 3x + c = 0$ does not have two distinct solutions in $[0, 1]$, where c is a constant.

5. Use the mean value theorem to prove the following inequality

$$\frac{x}{1+x} < \ln(1+x) < x, \text{ where } x > 0.$$

6. Find the specific solution for the following differential equations.

$$\dot{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix}$$

7. Consider the matrix

$$A = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

(a) Compute eigenvalues and eigenvectors.

(b) Diagonalize the matrix.

(c) Suppose that economic agents are either unemployed or employed. If an agent is unemployed, the probability that he will be unemployed next period is 0.8. The probability that he will be employed is 0.2. If an agent is employed the probability that he will be unemployed next period is 0.5. The probability that he will be employed is 0.5. Compute long run employment outcomes.

8. Assume that A and B are two symmetric $k \times k$ matrices.

(a) Show that

$$AB = (BA)^T$$

(b) Show that if A is invertible, then A^{-1} is also a symmetric matrix.

(c) Assume that we can write

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

with A_1 a 2×2, A_2 an $m \times 3$, A_3 a $3 \times n$ and A_4 a $p \times q$ matrix. What can we say about m, n, p, q? Which of the four submatrices is symmetric? Can you say something about the relationship between A_2 and A_3?

9. Suppose there are three states of the world tomorrow. Three securities exist that you can trade. The first security pays $v_1 = (1, 0, 3)^T$ the second security pays $v_2 = (0, 5, 6)^T$, the third pays $v_3 = (7, 8, c)^T$, where c is a real number, in states $(1, 2, 3)^T$ respectively.

(a) For what values of c can you generate any desired income stream by choosing an appropriate portfolio?

(b) If $c = 2$, what portfolio generates the income stream $(1, 1, 1)$?

(c) If $c = 30.6$, show that one security is redundant in the economy, i.e. for some i, $v_i \in span\,\langle v_j, v_k \rangle$.

(TURN OVER)

SECTION B

10. Suppose

$$Y_i = x_i b + Z_i,$$

where $Z_1, ..., Z_n$ are independent and identically distributed random variables, where each element Z_i $(i = 1, ..., n)$ is normally distributed with mean zero and variance σ^2, which is known. Recall that the probability density function of a normal random variable, s, with mean μ and variance σ^2 is given by

$$f\left(s; \mu, \sigma^2\right) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(s-\mu)^2}{2\sigma^2}\right\}.$$

Suppose $x_1, ..., x_n$ are fixed constants such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^2 = \lambda,$$

and clearly

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right)^{-1} = \lambda^{-1}.$$

(a) Write down the loglikelihood when the unknown parameter is β.

(b) Find the score and the maximum likelihood estimator (MLE) for β.

(c) Find the information matrix (in this case it is a scalar) and state the asymptotic distribution of the MLE once it is properly standardised.

(d) In this specific problem the MLE is unbiased. Is it possible to find a more efficient estimator in this case?

11. In a research programme on human health risk from contaminated river water $y = 10$ out of $n = 174$ one litre samples were identified as posing a significant risk to health.

(a) What is the distribution of y, the number of samples containing a health risk?

(TURN OVER for continuation of Question 11)

(b) Let π be the true probability that a one-litre sample represents a risk. Using a Beta(1,4) prior for π, find the posterior distribution for π given y.

Note: i)

Definition 1. *A random variable $X \sim Beta(\alpha, \beta)$ if its probability density function has the form*

$$f_X(x) \propto \left\{ \begin{array}{ll} x^{\alpha-1}(1-x)^{\beta-1} & 0 \leq x \leq 1 \\ 0 & else \end{array} \right\}$$

ii) *The second moment about the origin is given by*

$$\alpha(\alpha+1)/(\alpha+\beta+1)(\alpha+\beta)$$

(c) Summarise the posterior distribution by its first two moments.

(d) Find a normal approximation to the posterior distribution, and compute a 95% Bayesian confidence interval using this approximation.

12. Consider the problem

$$\max_{x,y} \quad -x^2 - xy - y^2$$

subject to

$$\begin{aligned} x - 2y &\leq -1 \\ 2x + y &\leq 2 \end{aligned}$$

(a) Are the Lagrangian first order conditions sufficient for this problem?

(b) Solve explicitly for the optimal x and y.

13. Show that if $f(x)$ is homogeneous of degree m and $g(x)$ is homogeneous of degree k, then $f(x^*) = \frac{\lambda kc}{m}$, where x^* attains the maximum of $f(x)$ subject to the constraint $g(x) = c$ and λ is the associated Lagrangian multiplier, where x is an n dimensional vector.

(TURN OVER)

14. Consider an open Leontief system with three fixed-proportions production sectors. Assume that the vector of labour requirements is $a_0^T = (1, 1, 1)^T$, i.e. 1 unit of labour is needed to produce 1 unit of each output. The input-output matrix is

$$A = \begin{bmatrix} 0.3 & 0 & 0.4 \\ 0.6 & 0.1 & 0 \\ 0 & 0.2 & 0.5 \end{bmatrix}$$

Assume that the wage rate is $w = 1$ and call $p^T = (p_1, p_2, p_3)^T$ the vector of output prices.

(a) Explain what an element a_{ij} of the matrix A represents.

(b) If the consumer demand for outputs is $c^T = (290 \text{ million}, 580 \text{ million}, 145 \text{ million})^T$, how much of each good must the economy produce?

(c) Write down the firms' zero-profit conditions in matrix format.

(d) What are the output prices?

15. Let A be an $n \times m$ matrix. Explain and prove the following two statements

(a) The set V of solutions to the system of equations $Ax = 0$ is a subspace of $\mathbb{R}^m$.

(b) The solution set of $Ax = b$ is the affine subspace $x_0 + Null(A)$, where x_0 is a particular solution to $Ax = b$ and $Null(A)$ is the Nullspace of A.

Let

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 3 & 1 & 3 \\ 1 & 0 & 0 \\ 4 & 8 & 1 \end{bmatrix}$$

(c) Find a basis for $Null(A)$ and $Col(A)$.

(d) Find a basis for the following set $B = \{b \in \mathbb{R}^4 \mid Ax = b \text{ has no solution}\}$.

END OF PAPER