Section A

1. Consider the following IS-LM model: Equilibrium in the goods sector is given by the equation

$$Y = C + I + G,$$

where Y is national income, C is consumer spending, I is investment and G is government spending. We assume that consumer spending is related to disposable income $Y - T$, where T are taxes, and the interest rate r:

$$C = c_0 + c_1(Y - T) - c_2 r,$$

with $c_1, c_2 > 0$. Taxes and investment are given by the equations

$$\begin{aligned} T &= t_0 + t_1 Y \\ I &= I^0 + a_0 Y - ar \end{aligned}$$

with $t_0, t_1, a_0, a > 0$.

Equilibrium in the money market is given by the equation

$$M_s = M^0 + mY - hr,$$

where the left-hand-side is the fixed money supply and the right-hand-side is the money demand in the economy.

   (a) Use Cramer's rule to solve this system of linear equations for the level of national income and the interest rate in terms of the exogenous variables.

   (b) Show that an increase in government spending will increase national income but also the interest rate.

2. Consider the following system of linear equations

$$\begin{aligned} 2x + 3y - tz &= 1 \\ x - sy + 3z &= 4. \end{aligned}$$

   (a) How many solutions do you expect this system of linear equations to have? Explain your answer.

   (b) For which values of t and s will there be no solution? For which values will there be a solution?

   (c) For $t = -6$ and $s = 1$, provide a solution to the system of linear equations.

3. Answer both parts of this question.

    (a) Let A be an $n \times n$ matrix. Prove that r is an eigenvalue of A if and only if there exists a non-zero vector v such that $(A - rI)v = 0$.

    (b) The probability that an economy is in an expansionary phase today given it was also in an expansionary phase in the previous period is 0.7. The probability that an economy is in recession today given it was also in recession in the previous period is 0.8. Compute the long-run probabilities of expansion and recession.

4. Answer both parts of this question.

    (a) Suppose that $f,\ g^1, .., g^k$ are C^1 functions of n variables. Suppose that $x^* \in \mathbb{R}^n$ maximizes f subject to $g^1(x) \leq b_1, .., g^k(x) \leq b_k$. Let $\lambda^* \in \mathbb{R}^k$ denote the associated Lagrangian multiplier, and let $L(x^*, \lambda^*)$ denote the value of the Lagrangian evaluated at (x^*, λ^*). Show that $f(x^*) = L(x^*, \lambda^*)$.

    (b) Let $f(x)$ be a function defined on $\mathbb{R}$. Show that $f'(x) = 0$ for all x if and only if f is a constant function.

5. Consider the following function defined on $\mathbb{R}^2$.

$$f(x,y) = \begin{cases} \frac{xy}{x^{\frac{3}{2}} + y^3}, & \text{if } x > 0,\ y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

    (a) Is f continuous at the point $(0,0)$? Explain your answer.

    (b) Does f have partial derivatives at $(0,0)$? Explain your answer.

6. Find the specific solution to the following differential equations. In each case, describe what happens to the dependent variable(s) as $t \longrightarrow \infty$.

    (a)

$$\dot{y} + t^2(y - 7) = 0;\ y(1) = 8.$$

    (b)

$$\dot{x} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} x;\ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(TURN OVER)

7. Consider a random variable T with probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } 0 < t < \infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the cumulative distribution function for T and show that

$$\Pr(T > r + s | T > r) = \Pr(T > s)$$

(b) You are given data on a random sample T_i, $i = 1, .., n$, where T_i represents the time to re-enter the labour force for individual i after becoming unemployed. Comment on the adequacy of the density function $f(t)$ to represent this process.

8. X and Y are independent discrete random variables with range $\{0, 1, 2, ..., n\}$ and binomial distribution

$$p_X(j) = p_Y(j) = \binom{n}{j} p^j q^{n-j}$$

where $q = 1 - p$.

A and B are independent discrete random variables with non-negative integers $\{0, 1, 2, ..., n\}$ as range, and with geometric distribution function

$$p_A(j) = p_B(j) = q^j p$$

(a) Find the moment generating function of X and the moment generating function of A.

(b) Find the distribution of $Z = X + Y$.

(c) Find the moment generating function of $C = A + B$.

9. Suppose $X_1, ..., X_n$ is a random sample where $\mu := \mathbb{E}X_i$ and $\sigma^2 := Var(X_i)$ are unknown. Define

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i,$$

$$S^2 := \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$$

and

$$\tilde{S}^2 := \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2.$$

(a) Compute the bias of S^2 and $\tilde{S}^2$. Are both estimators asymptotically unbiased?

(b) Verify that $S^2 = \left(\frac{n-1}{n}\right)\tilde{S}^2$ and compute the variance of S^2 assuming that $Var\left(\tilde{S}^2\right) = 2\sigma^4/(n-1)$.

(c) Compute the mean square error (MSE) of S^2 and $\tilde{S}^2$ and find which estimator is to be preferred in terms of MSE when $\sigma^2 = 1$ and $n = 20$.

(TURN OVER)

Section B

10. Let A be an $n \times n$ nonsingular matrix.

    (a) Show that
    $$A^{-1} = \frac{1}{\det A} \cdot adj\,(A)\,.$$
    Hint: Show first that $A \cdot Adj(A) = \det A \cdot I$ and then use this identity to prove the statement.

    (b) Use the adjoint of A to compute A^{-1} for
    $$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

11. Answer both parts of this question.

    (a) Let V be a subspace of $\mathbb{R}^n$.

    i. Let V be of dimension m. Prove that, if $k > m$, any set of k vectors in V must be linearly dependent.

    ii. Prove that any two bases for V must contain the same number of vectors.

    (b) Consider the matrix
    $$A = \begin{pmatrix} 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 3 & -1 & 1 \\ 1 & -1 & 6 & 7 & 2 \end{pmatrix}.$$

    i. Find a basis for $Col(A)$.

    ii. What are the dimensions of $Col(A)$ and $Null(A)$?

    iii. Find a basis for $Null(A)$.

12. Consider the problem
$$\min_{x,\ y}\ 2x^2 + 2y^2 - 2xy - 9y$$
subject to
$$\begin{aligned} 4x + 3y &\leq 10 \\ 4x^2 - y &\leq 2 \end{aligned}$$

    (a) Write down the Lagrangian and the first order conditions for the problem.

    (b) Are the first order conditions sufficient for the solution to be globally optimal? Explain your answer.

    (c) Is it possible that at the optimal solution, neither constraint is binding? Explain your answer.

    (d) Find the optimal solution to the problem.

13. Let $g(x)$ be a function defined over the unit interval $[0,1]$. Suppose that $g(0) = g(1) = 0$, and $g''(x) \leq 0$, for any $x \in [0,1]$.

    (a) Show that $g(x) \geq 0$, for any $x \in [0,1]$.

    (b) Show that for any x_1, x_2, ..., x_n such that $0 < x_1 < x_2 < .. < x_n < 1$, and for any λ_1, λ_2,..., λ_n such that $\lambda_i \geq 0$, for all i, and $\sum_{i=1}^{n} \lambda_i = 1$, $g\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i g(x_i)$.

    (c) Show that for any $x \in (0,1)$, $g'(x) \leq \frac{g(x)}{x}$.

14. A random sample consists of n independent and identically distributed random variables $X_i, i = 1, ..., n$, where

$$X \sim f(x;\theta) = \theta^x (1-\theta)^{1-x}, \quad 0 \leq \theta \leq 1, \quad x = 0, 1$$

    (a) Find the distribution of the sample and the log-likelihood function.

    (b) Consider the following statistics

    i) $\widehat{\theta}_1 = X_1$ ii) $\widehat{\theta}_2 = \frac{1}{2}(X_1 + X_2)$

    iii) $\widehat{\theta}_3 = \frac{1}{n}\sum_{i=1}^{n} X_i$ iv) $\widehat{\theta}_4 = (X_1 - X_n)$

    v) $\widehat{\theta}_5 = \frac{1}{n}\sum_{i=1}^{n} X_i^{\alpha}$

    Which of the above random variables constitutes a possible estimator of θ?

    (c) Find an expression for Fisher information for X_i and generalise this to an i.i.d. sample. Find the Cramer Rao lower bound, and thereby determine which estimator is fully efficient.

    (d) Find the mean square error for $\widehat{\theta}_2$ and $\widehat{\theta}_3$ in (b) and in doing so demonstrate which estimator is consistent for θ.

(TURN OVER)

15. Suppose that for $i = 1, ..., n$,

$$Y_i = \alpha_0 + \beta_0 x_i + \varepsilon_i$$

where $x_1, ..., x_n$ are deterministic and $\varepsilon_1, ..., \varepsilon_n$ are independent and identically distributed normal random variables with mean zero and variance σ^2. Recall that a random variable Z with mean μ and variance σ^2 has density function $pdf\,(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(z-\mu)^2\right\}$.

(a) Write down the log-likelihood to estimate (α_0, β_0).

(b) Define the maximum likelihood estimator (MLE) for this problem (do not need to compute it) and show that it must be equivalent to the one found by ordinary least square (hint: you only need to compare the objective functions).

(c) Suppose you know that $\alpha_0 = 0$. Compute the MLE of β_0 and show that it is unbiased.

(d) Use Chebyshev's inequality to show that $n^{-1}\sum_{i=1}^{n}\varepsilon_i \to 0$ in probability. Hence deduce consistency of the MLE of β_0 when $x_i = 1$ for all $i = 1, ..., n$.

END OF PAPER