ECONOMICS TRIPOS PART IIA

Thursday 5 June 2008 9-12

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections; A and B.

Each Section carries 50% of the total marks.

Candidates should attempt ***SIX*** *questions from Section A and* ***THREE*** *questions from Section B.*

Write on **one** *side of the paper only.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 2	Approved calculators allowed
Rough Work pads	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. (a) Consider an $n \times n$ matrix A where $n(n-1)$ of its elements are 0 and the remaining n elements are not 0. Are the following two statements true or false?

   (i) A's determinant must be equal to 0.

   (ii) A's determinant must not be equal to 0.

   (b) For what value of r is the following determinant 0? $\det \begin{vmatrix} 1 & 2 & 0 & r \\ -2 & 3 & 1 & 2 \\ 1 & 2 & 0 & 4 \\ 1 & 1 & 1 & 0 \end{vmatrix}$

2. (a) For 3 vectors $v_1,\ v_2,\ v_3 \in \mathbb{R}^n$, is it possible that $\{v_1, v_2\}$ are linearly independent and that $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are linearly dependent? If yes, give an example, if no, prove it.

   (b) Show that if a collection of vectors $\{v_1, ..., v_k\}$ are linearly independent then any subset of those vectors are also linearly independent.

3. (a) Assume that A is a symmetric matrix with

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

   Show that a necessary condition for A to be positive definite is that $a > 0$ and $c > 0$.

   (b) Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ is indefinite.

4. (a) Prove that if $f(x)$ is differentiable at ξ, then $f(x)$ must be continuous at ξ. Give an example to show that the converse is not necessarily true.

   (b) Prove that the derivative of $f(x)/g(x)$ evaluated at ξ is $\frac{f'(\xi)g(\xi)-f(\xi)g'(\xi)}{g^2(\xi)}$, provided that $f'(\xi)$ and $g'(\xi)$ exist and, $g(\xi)$ is nonzero.

   (c) Examine the continuity and differentiability of the function $f(x) = x^n \sin(1/x)$ for $n = 0, 1, 2, 3$.

5. An investment costing k produces a stream of one unit of output every year for T years, using inputs costing c per unit, after which time it suddenly fails and becomes worthless. The rate of interest is r , time is continuous, and the market is large (in numbers of units bought) and competitive.

   (a) Find an expression for the market clearing price, p.

   (b) Find a formula describing the value of the investment after t years, V_t. Show V satisfies $p - c = rV_t - dV_t/dt$.

   (c) (c) Compare the initial economic rate of depreciation $-dV_t/dt$ with that of conventional accounting, which writes down the capital value each year by k/T, when $T = 10$ years and $r = 10\%$.

6. An economy evolves according to the following equations

$$\begin{aligned} Y_t &= F(K_t, L_t) = L_t f(k_t), \qquad k_t = K_t/L_t, \\ \frac{dK}{dt} &= sY_t - \delta K_t,\ s, \delta > 0,\ f(k) > kf'(k), \end{aligned}$$

where Y_t is GDP, K_t is capital stock, and L_t is the labour force, all at date t. The rate of net capital formation, dK/dt, is equal to savings, sY_t, less depreciation δK_t, where the capital stock evaporates at rate δ. The labour force grows at a proportional rate that depends on the standard of living, y_t, and will be falling if people are starving, and at a sufficiently high standard of living the population (and labour force) will stabilise and grow no further:

$$\begin{aligned} \frac{dL}{dt} &= L_t g(y_t), \quad y_t = Y_t/L_t = f(k_t), \\ g(0) &< 0, \quad g'(f(k)) > 0,\ k < k^*, \\ g(f(k)) &= 0,\ k \geq k^{**} > k^*. \end{aligned}$$

(TURN OVER for Continuation of Question 6)

(a) Find the equation of motion of k in the form $\frac{dk}{dt} = \phi(k)$.

(b) You are told that the equation

$$\frac{sf(k)}{k} = \delta + g(f(k))$$

has three solutions, $k_1 < k^* < k_2 < k^{**} < k_3$. Show that k_1 and k_3 are stable equilibria and give reasons why k_2 might be expected to be unstable.

7. A probability function Pr(.) is defined as a real-valued set function on the class of all subsets of the sample space Ω; where Ω has k elements and k is finite. The value associated with a subset A is denoted $\Pr(A)$. The assignment of probabilities must satisfy the following three axioms:

(i) $\Pr(\Omega) = 1$

(ii) $\Pr(A) \geq 0$ for all $A \subset \Omega$

(iii) $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ if $A \cap B = \varnothing$

Prove the following result:
Let $n(A), n(\Omega)$ denote, respectively the number of elements in A and Ω. The probability function $\Pr(A) := \frac{n(A)}{n(\Omega)}$ satisfies the three axioms listed above.

8. Assume that observations on a discrete process are observed for a fixed time period of length t, where t is any positive number. The number of events that occur in this fixed interval $(0, t]$ is a random variable G, where the range of G is discrete. Given that the assumptions of a Poisson process hold, we may subdivide our interval of length t into $n = t/\Delta t$ nonoverlapping, equal length sections.

(a) Find an expression for the probability distribution of G.

(b) Letting p_G denote the probability distribution of G, find an expression for the limit of p_G as $\Delta t \to 0$ and thus $n \to \infty$.

9. Suppose that in a test, the null hypothesis is $H_0 : \theta = 0$ and the alternative is $H_1 : \theta = 2$. Suppose that you use a statistic $\hat{\theta}$ that is asymptotically normal with mean θ and variance 1.

   (a) Find expressions for the type I error and for the power function of such test for a given critical region C_α with asymptotic nominal size equal to α.

   (b) Using the result in (a), show that the power is largest if C_α is of the form $C_\alpha := [c_\alpha, \infty)$ for suitably chosen $c_\alpha > 0$ (i.e. a one sided test on the right).

(TURN OVER)

SECTION B

10. Suppose that uncertainty is represented by S states of the world. A trader has a stochastic endowment of income over these states given by the vector $w \in \mathbb{R}^S$. He wishes to hedge against risk. He can trade J securities. The payoff of security j in state s is V_{sj}. Payoffs are collected in the following payoff matrix:

$$V = \begin{bmatrix} V_{11} & .. & V_{1J} \\ .. & .. & .. \\ V_{S1} & .. & V_{SJ} \end{bmatrix}$$

A portfolio of securities is $z \in \mathbb{R}^J$ with return Vz and the trader's income stream $y \in \mathbb{R}^S$ given endowment w and portfolio z is $y = w + Vz$.

(a) (i) What is the minimum number of securities necessary to generate any income stream? Explain under what condition(s) on V can he generate *any* income stream $y \in \mathbb{R}^S$?

(ii) A portfolio is called *duplicable* if there exists a different portfolio with exactly the same return. Under what condition(s) on V is any portfolio duplicable?

(b) Assume that $w = (2, 1, 0, -1)^T$ and that

$$V = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & 3 & -5 \\ 2 & -4 & -2 & 2 \\ 0 & 1 & 3 & -2 \end{bmatrix}.$$

(i) Find all income streams that can be generated by this income and set of securities. Verify that a riskless income stream can be generated and calculate the corresponding portfolio weights.

(ii) Find all portfolios that can be duplicated.

11. In an economy at time t income is denoted by y_t, savings are denoted by s_t and consumption is denoted by c_t. The evolution of the three variables over time is given by the syste of linear difference equations.

$$\begin{array}{rclcrcrcr} y_t & = & 5y_{t-1} & + & 2s_{t-1} & + & 4c_{t-1} \\ s_t & = & -3y_{t-1} & + & 6s_{t-1} & + & 2c_{t-1} \\ c_t & = & 3y_{t-1} & - & 3s_{t-1} & + & c_{t-1} \end{array}$$

(a) Write down the coordinate matrix corresponding to the above system of linear difference equations and find its characteristic polynomial.

(b) Using the fact that one eigenvalue of the coordinate matrix is $r_1 = 2.0$, find the other two eigenvalues

(c) Find the three corresponding eigenvectors.

(d) Find expressions for (y_t, s_t, c_t) as a function of initial conditions (y_0, s_0, c_0). What will long-run income, savings and consumption be in this economy?

12. Car owning commuters live in a village 3 km from Cambridge. Half the commuting population is willing and able to cycle (the remainder never would but are otherwise identical). Commuters have an inelastic demand to travel but half can choose how to travel, and they leave at the rate of 1,600/hour in the morning. The generalized cost of travel for cars is $c = 20 + 500/v$ pence per km, where v is the car's speed in km per hour and £5 is the perceived time cost per hour in a car. Similarly for cycles the cost is $c_c = 5 + 600/v_c$ where $v_c = 15$ is the cycle's speed in km per hour and the perceived time cost of cycling is £6/hr. The car speed depends on traffic, q vehicles/lane-mile/hr. as

$$v = 50 - \beta q/k, \qquad \beta = 0.035,$$

where k is the capacity of the road (number of lanes) initially $k = 1$. Cycle speeds are independent of traffic volume. Total social cost of commuting is $C = qc + nc_c$ per km per hour where n is the number of cycles/hr.

(TURN OVER for Continuation of Question 12)

(a) Find the equilibrium volume of car and cycle traffic and show that the number of cyclists is less than the number willing to cycle.

(b) What is the marginal social cost (MSC) and private cost (PC) of motoring at this equilibrium?

(c) Suppose cars must pay a charge per km equal to the difference between the MSC and PC. What would the new equilibrium number of cars be if all commuters were willing to cycle and what would be the level of the toll?

(d) What is the equilibrium volume of car and cycle traffic and the difference between the MSC and PC given the actual number of willing cyclists? Would it matter if the toll were set at the answer to (c)?

13. The cost function of a firm is defined as $C(\boldsymbol{w}, y) = \underset{\boldsymbol{x}}{Min} \boldsymbol{w'x}$ s.t. $f(\boldsymbol{x}) \geq y$ where y is output, $f(\boldsymbol{x})$ is the production function, $\boldsymbol{x}$ is the vector of factor inputs, and $\boldsymbol{w}$ is the vector of corresponding factor prices. $C(\boldsymbol{w}, y)$ and $f(\boldsymbol{x})$ are differentiable functions.

(a) Prove that $C(\boldsymbol{w}, y)$ is homogenous of degree 1 and concave in $\boldsymbol{w}$.

(b) Prove Shephard's Lemma that $\partial C(\boldsymbol{w}, y)/\partial w_i = x_i$.

(c) Suppose that $y = K^\alpha L^{1-\alpha}$ with $\alpha, \beta > 0$, and the factor prices of K, L are r, w. Find the cost function and confirm that $\partial C(w, r, y)/\partial r = K$.

14. A manufacturing process produces motor vehicles in a given lot size. Among the lots there is variation in the percentage of defective vehicles, with p, the fraction of defective vehicles, given in Table B1. A random sample of 10 vehicles is chosen from the lots and after testing it is found that one of the vehicles is defective

(a) Find the likelihood of obtaining one defective vehicle in a sample of 10 for each value of p.

(b) Find the posterior probability distribution of the fraction of defective vehicles.

(TURN OVER for Continuation of Question 14)

Table B1

Fraction Defective p	Prior Probability function P(p)
.01	.25
.02	.35
.03	.20
.06	.15
.10	.04
.20	.01
	1.00

15. Suppose that $X_1, ..., X_n$ is a random sample from a Poisson distribution with mean arrival rate λ and probability mass function

$$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, ..$$

(TURN OVER for Continuation of Question 15)

(a) Write down the log-likelihood function for this problem.

(b) Find the maximum likelihood estimator (MLE) and give an expression for its variance (Hint: use the property of MLE to find its variance).

(c) Suppose that $Y_1, ..., Y_n$ is some random sample from some distribution with mean zero and variance one, such that

$$\begin{aligned} Cov\,(Y_i, X_j) &= -2/3 \text{ if } i = j \\ &= 0 \text{ otherwise.} \end{aligned}$$

Consider the following estimator for λ:

$$\tilde{\lambda} := \frac{1}{n} \sum_{i=1}^{n} (Y_i + X_i)\,.$$

Compute the mean and variance of this estimator.

(d) How does $\tilde{\lambda}$ compare to the MLE in terms of efficiency?. Would you prefer $\tilde{\lambda}$ to the MLE?

END OF PAPER