ECONOMICS TRIPOS PART IIA

Thursday 4 June 2009 9-12

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections: A and B.

Each Section carries 50% of the total marks.

Candidates should attempt ***SIX*** *questions from Section A and* ***THREE*** *questions from Section B.*

Write on **one** *side of the paper only.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 2	Approved calculators allowed
Rough Work pads	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

where $a_{11}a_{22}a_{33} \neq 0$. Prove that A is non-singular.

2. Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and vector $b \in \mathbb{R}^2$. Suppose the equation system

$$Ax = b$$

has two solutions denoted x' and x''. Are there more solutions to this system? If yes, show how they may be constructed. If not, explain why not.

3. Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \text{ and } v_3 = \begin{pmatrix} 9 \\ 0 \\ 9 \end{pmatrix}.$$

Do these vectors form a basis for $\mathbb{R}^3$? Explain or prove your answer.

4. Consider the following result.

Theorem 1 *For a convex function $g(\cdot)$ and a continuous random variable X with probability density function $f(\cdot)$, the following holds.*

$$E(g(X)) = \int_x g(x)f(x)dx \geq g(\int_x xf(x)dx) \geq g(E(X)).$$

If $g(\cdot)$ is a concave function then

$$E(g(X)) \leq g(E(X)).$$

Using this theorem show why the predictions of y, namely $\hat{y}_i = \exp(\ln \hat{y}_i)$, when β is estimated using the log-linear regression

$$\ln y_i = \mathbf{x}_i'\beta + \varepsilon_i$$

are biased, where $\mathbf{x}_i$ denotes a vector of exogenous variables, β is a vector of unknown parameters, and $\varepsilon_i \sim i.i.d.\ (0, \sigma^2)$.

5. Let $M_X(t)$ denote the moment generating function of a random variable X and f_X the corresponding probability mass function.

   (a) Suppose that X has range $\{1, 2, 3, ..., n\}$ and $f_X(j) = 1/n$ for $1 \leq j \leq n$. Show that

   $$M_X(t) = \frac{e^t(e^{nt} - 1)}{n(e^t - 1)}.$$

   (b) Suppose that X has range $\{0, 1, 2, 3, ..., n\}$ and

   $$M_X(t) = (pe^t + q)^n.$$

   Show that the variance of X is given by $np(1 - p)$.

(TURN OVER.)

6. In a research program on human health risk from recreational contact with water contaminated by pathogenic microbiological material, water samples were taken from sites identified as having a heavy environmental impact from seagulls and waterfowl. Out of $n = 116$ one-litre water samples $y = 17$ samples contained the pathogenic material.

   (a) What is the distribution of y, the number of samples containing the pathogenic material?

   (b) Let π be the true probability that a one-litre water sample from this type of site contains pathogenic material. Using a $Beta(1, 4)$ prior for π. Find the posterior distribution of π given y.

   [**Note.** **(i)** A random variable $X \sim Beta(\alpha, \beta)$ if its probability density function has the form

$$f_X(x) \propto \begin{cases} x^{\alpha-1}(1-x)^{\beta-1} & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

   (ii) The second moment about the origin is given by $\frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}$.]

7. You are given the following linear regression model.

$$y = \beta x^* + u$$

   Unfortunately, you do not observe x^*, instead you observe

$$x = x^* + \varepsilon.$$

   Assume that

$$plim(\frac{x^{*\prime}u}{N}) = plim(\frac{x^{*\prime}\varepsilon}{N}) = plim(\frac{u'\varepsilon}{N}) = 0.$$

   Define $plim(\frac{x^{*\prime}x^*}{N}) = \sigma^2_{x^*}$, $plim(\frac{u'u}{N}) = \sigma^2_u$ and $plim(\frac{\varepsilon'\varepsilon}{N}) = \sigma^2_\varepsilon$ (all finite and positive). Show that the least squares estimators of b and a from the regressions $y = bx + e$ and $x = ay + \nu$, respectively, obey the relation

$$plim(b) < \beta < plim(1/a) \text{ if } \beta > 0.$$

8. For the technology $y = f(z_1, z_2)$ where y is output and z_i $(i = 1, 2)$ are inputs, the cost function takes the following form

$$C(w_1, w_2, y) = e^{y/2}\sqrt{w_1 w_2}$$

where w_i $(i = 1, 2)$ are input prices.

(a) Show that this cost function is homogenous of degree 1 and non-decreasing in prices.

(b) Find the supply function.

9. Assume that demand Q_{dt} and supply Q_{st} obey

$$\begin{aligned} Q_{dt} &= c + zP_t \\ Q_{st} &= g + hP_t \end{aligned}$$

and price P_t evolves according to

$$P_{t+1} = P_t - a(Q_{st} - Q_{dt})$$

i.e., price is determined by the level of inventory $(Q_{st} - Q_{dt})$. Also, assume that $a > 0$ so that a buildup in inventory $(Q_{st} > Q_{dt})$ will tend to reduce price and a depletion of inventory $(Q_{st} < Q_{dt})$ will cause price to rise.

(a) Solve for the price P_t.

(b) Comment on the stability conditions of the time path.

10. Suppose that f and g are inverse functions and that both are twice differentiable, i.e. $g(f(x)) = x$ for all x.

(a) Derive an expression for $g''(f(x))$.

(b) Show that if f is strictly increasing and convex, then g is concave.

(TURN OVER.)

SECTION B

11. Answer **both** parts.

(a) Consider the following closed-economy IS-LM model. Total national income (and spending) is denoted by Y, consumer spending is denoted by C with $C = bY - cr$, where $b \in (0, 1)$ and $c > 0$. The interest rate is denoted by r. Government spending is G and the investment demand from firms is $I^d = I^0 - ar$, where $a > 0$. Desired national saving is S^d. Money supply M^s is fixed and the demand for money M^d obeys $M^d = M^0 + mY - hr$, where $m > 0$ and $h > 0$.

i. Derive the IS curve depicting r as a function of Y when desired national saving meets investment demand. Also derive the LM curve relating r at which the money market clears for a given level of output.

ii. Obtain the system of equations which describes equilibrium in both markets. Assume that $b = m = 0.5, a = c = 1$, and $h = 2$ and solve for the equilibrium values of Y and r when $M^s = 100, M^0 = 50, I^0 = 200, G = 50$.

(b) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

denote the 3×3 coefficient matrix of the system of equations

$$Ax = b$$

for a vector $b \in \mathbb{R}^3$ and B denote the augmented matrix that represents this system. Let C denote the 3×4 matrix obtained from B after multiplying the last row of B by the scalar λ and adding the result to the first row, i.e. C differs from B only in the last row.

i. Show that if $x' \in \mathbb{R}^3$ is a solution to the system of equations represented by matrix B, then it is also a solution to the system represented by matrix C.

ii. Show that if $x'' \in \mathbb{R}^3$ is a solution to the system of equations represented by matrix C, then it is also a solution to the system represented by matrix B.

12. Answer **both** parts.

(a) Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

be a symmetric ($a_{12} = a_{21}$) non-singular matrix.

i. Let r be an eigenvalue of A. Show that $r \neq 0$ and that $1/r$ is an eigenvalue of A^{-1}.

ii. Let v be an eigenvector of A corresponding to r. Show that v is also an eigenvector of A^{-1}.

(b) Let

$$\hat{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

i. Find the eigenvalues of $\hat{A}$ and derive an eigenvector with length (or norm) equal to 1 corresponding to each eigenvalue.

ii. Show that for any positive integer n,

$$\hat{A}^n = \begin{bmatrix} \frac{3^n+1}{2} & \frac{3^n-1}{2} \\ \frac{3^n-1}{2} & \frac{3^n+1}{2} \end{bmatrix}.$$

13. Let E, C, and M, represent three binary random variables denoting, respectively, recovery from an illness, whether a drug was taken, and gender. Eighty individuals suffering from a given disease were randomly sampled from a population of sufferers. Table 1 represents some summary statistics on the joint distribution $f(E, C, M)$ by treatment, recovery and gender.

(a) We observe that

$$\begin{aligned} \widehat{\mu}_{E|C} &> \widehat{\mu}_{E|\bar{C}} \\ \widehat{\mu}_{E|C,M} &< \widehat{\mu}_{E|\bar{C},M} \\ \widehat{\mu}_{E|C,\bar{M}} &< \widehat{\mu}_{E|\bar{C},\bar{M}} \end{aligned}$$

where $\widehat{\mu}_{E|C} > \widehat{\mu}_{E|\bar{C}}$ indicates that the mean recovery rate for patients taking the drugs exceeds that of the control. Explain this outcome and comment on the observation that this result is a paradox.

(TURN OVER for Table 1 and continuation of question 13.)

(b) The joint distribution of the two random variables E and C may be written as

$$f_{E,C}(e, c) = f_{E|C}(e|c)f_C(c) \quad (1)$$

By considering this joint distribution in conjunction with the full distribution $f(E, C, M)$ demonstrate that the paradox of (a) arises from an endogeneity problem.

(c) An analyst has access to the data on the eighty individuals that were used in the study. In examining the relationship between recovery and treatment he specifies the following regression equation.

$$E_i = \alpha + \beta C_i + \varepsilon_i, \quad (2)$$

where α and β are unknown parameters and $\varepsilon_i \sim i.i.d(0, \sigma^2)$. In the light of your discussion in (b), comment on the appropriateness of this regression and how (2) might modified to test for endogeneity.

Table 1: Combined (Males and Females)

	E	$\bar{E}$	Total
Drug (C)	20	20	40
$\bar{C}$	16	24	40

Males

	E	$\bar{E}$	Total
Drug (C)	18	12	30
$\bar{C}$	7	3	10

Females

	E	$\bar{E}$	Total
Drug (C)	2	8	10
$\bar{C}$	9	21	30

14. Consider the following linear model.

$$y = X\beta + \varepsilon$$

where X is an $n \times k$ matrix and y is a $n \times 1$ vector.

(a) Write down the likelihood function assuming that $\varepsilon \sim N(0, \sigma_\varepsilon^2 I_n)$, where I_n denotes the $n \times n$ identity matrix.

(b) Derive the maximum likelihood estimator of β by concentrating out the variance of ε (σ_ε^2).

(c) How would you carry out this estimation in practice.

(d) Find the variance of this estimator and compare it to that of the least squares estimator. Comment on your derivation.

15. Consider the following utility function.

$$U = \ln x_1 + x_2$$

Let income be denoted by y, price of good 1 by p_1, and price of good 2 by p_2.

(a) Find the Marshallian and Hicksian demands, the expenditure function, and the indirect utility function.

(b) Show that the indirect utility function is homogenous of degree zero in prices and income.

(c) Establish the degree of homogeneity of the marginal utility of income.

(d) Derive the own-price elasticities of demand for x_1 and x_2.

(TURN OVER.)

16. A vertically integrated firm produces an input at zero cost. The firm can transform this input into a final product at a cost of $c > 0$ per unit and sell it to final consumers or it can sell the input to a downstream firm at a price w for transformation and sale at a cost of $m > 0$ per unit, where $m < c$. The demand for the final product is $q = a - p$, where $a > c$ and p is the price of the final product.

    (a) Find the expression for the maximum profit that the vertically integrated firm can make if it refuses to sell to the downstream entrant. [*Hint*: use p as the choice variable.]

    (b) Find the expression for maximum profit that the vertically integrated firm can make if it sells all its input to the downstream entrant. [*Hint*: use p and w as choice variables.] Under what conditions would it be more profitable for the vertically integrated firm to make and sell the product? Will this be the case if $a = 25$, $c = 7$, and $m = 5$?

    (c) Suppose that a regulator requires that the final product price is no higher than $P = \frac{1}{2}(a + c)$ and that this is lower than the price the downstream firm would charge if it buys from the upstream firm at w^*, the profit maximising choice for the upstream firm in part (b). Prove that the upstream firm will prefer to sell to the downstream firm under these conditions, rather than making and selling the product without the price control.

END OF PAPER.