ECONOMICS TRIPOS PART IIA

Thursday 3 June 2010 9-12

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections: A and B.

Each Section carries 50% of the total marks.

Section A is divided into three sub-sections and candidates are required to answer a total of ***SIX*** *questions, with at least one question from each sub-section.*

Section B is divided into three sub-sections and candidates are required to answer a total of ***THREE*** *questions, with one question from each sub-section.*

Write on **one** *side of the paper only.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 2	Approved calculators allowed
Rough Work pads	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Section A: Answer <u>six</u> questions from this section

Sub-section A.I: Answer <u>at least one question</u> from this sub-section

1. Let P be the set of all polynomial functions in an indeterminate $x \in \mathbb{R}_+$ with coefficients in $\mathbb{R}^3$.

   (a) Show that P is a vector space.

   (b) Consider the collection $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$, where $p_1(x) = x^2 + x + 1$, $p_2(x) = x^2 - x - 2$, $p_3(x) = x^2 + x - 1$, $p_4(x) = x - 1$. Show that this collection is linearly dependent.

   (c) What is the span of the collection $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$? [**Hint.** It is an infinite set, so rather than list all its elements, it would be more useful to provide a general description.]

2. Let $a > 0$ and $b_1, \ldots, b_n$ be real numbers, where $n \geq 2$. Consider the system of n equations given by

$$\begin{bmatrix} 1+a & a & \ldots & a \\ a & 1+a & \ldots & a \\ \vdots & \vdots & \vdots & \vdots \\ a & a & \ldots & 1+a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

   (a) Solve this system if $b_1 = b_2 = \ldots = b_n = b$.

   (b) Show that this system of equations has a unique solution for general values of $b_1, b_2, \ldots, b_n$, i.e. not necessarily equal.

3. Let A and B be two $n \times n$ matrices, where $n \geq 2$, and A is non-singular.

   (a) Suppose that for some real number $\beta \neq 0$ the matrix $A + \beta B$ is singular. Show that $(-1/\beta)$ is an eigenvalue for $A^{-1}B$.

   (b) Using the result in (a) and the theorem given below, show that there can be *at most* n real numbers $\beta_1, \ldots, \beta_n$ such that $A + \beta_j B$ is singular for each $j \in \{1, \ldots, n\}$.
   [**Theorem.** A polynomial of degree m can have at most m distinct roots.]

Sub-section A.II: Answer at least one question from this sub-section

4. Answer all parts.

   (a) Let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences of real numbers, where $x_n \to a$, $y_n \to b \neq 0$. Prove that the sequence $\{z_n\}_n$, where $z_n = (x_n/y_n)$, tends to the limit (a/b).

   (b) By setting $x = y^{-1}$ or otherwise, evaluate

$$\lim_{y \to \infty} [y - \sqrt{(1 + y^2)}].$$

5. Let $F(x, y) = (2x^{-1}y^{-2} + y^{-1}x^{-2})^{-\frac{1}{3}}$. Find the slope of the curve $F(x, y) = \frac{1}{2}$ when $y = 1$ and $x > 0$, using the implicit function theorem.

6. Suppose you run a wine company. As you age the wine it improves, so that the quality evolves according to

$$Q(t) = e^{t - 0.5Q_0 t^2}$$

where $Q(t)$ is quality, t is years and $Q_0 \neq 0$ is initial quality of the vintage.

   (a) Assuming an interest rate r, and that the price you can sell the wine at is directly proportional to its quality, calculate the optimal time to bring the wine to market.

   (b) In what sense do you sell your wine before its time?

TURN OVER

Sub-section A.III: Answer <u>at least one question</u> from this sub-section

7. Suppose

$$y_t = \beta x_t^* + \varepsilon_t$$

where $t = 1, 2, ..., n$, x_t^* is an unobserved variable measured imperfectly by

$$x_t = x_t^* + \eta_t.$$

ε_t and η_t are i.i.d. with mean zero and variances σ_ε^2 and σ_η^2 respectively. Suppose ε_t, η_t, and x_t^* are all mutually uncorrelated and $plim(x^{*\prime}x^*)/n$ is positive and finite.

(a) If b is the OLS estimate from regressing y_t on x_t, find $plim(b - \beta)$ and comment on the nature of the bias.

(b) Suppose now there is an instrumental variable (IV) w_t for x_t. Provide conditions such that the IV estimator

$$b_{IV} = (w'x)^{-1}w'y$$

is a consistent estimator of β and prove your answer.

8. Suppose we perform n independent hypothesis tests, each with a pre-chosen type one error, α. Consider the following two events.

(FP) "a false positive", denotes the event of rejecting a null hypothesis when it is true.

(AFP) "at least one false positive", denotes the event of rejecting *at least one* null hypothesis when it is true, over n tests,

(a) Find an expression for the distribution of "false positives" over the n independent tests.

(b) Find the probability of at least one false positive.

9. Y_i, $i = 1, ..., n$ denotes a sequence of i.i.d Bernoulli random variables with $E[Y_i] = p$ for all i. Show that

$$\frac{S_n}{n} \xrightarrow{P} p \text{ as } n \to \infty$$

where $S_n = \sum_{i=1}^{n} Y_i$ and $\xrightarrow{P}$ denotes convergence in probability.

Section B: Answer <u>three</u> questions from this section

Sub-section B.I: Answer <u>only one question</u> from this sub-section

10. Answer all parts.

(a) For what values of α and β does the system of equations given by

$$\begin{bmatrix} 2x + & 4y + & (\alpha+3)\,z \\ x + & 3y + & z \\ (\alpha-2)x + & 2y + & 3z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \beta \end{bmatrix}$$

have a unique solution.

(b) Let A and B be two $n \times n$ matrices, where $n \geq 3$, that have the same column spaces. Answer the following with an explanation or proof for each.

i. Do A and B have the same number of pivots and the same null space?

ii. Show that A is invertible if and only if B is invertible.

(c) Let A be an $n \times n$ matrix, where $n \geq 2$, with the following properties.

$$\begin{array}{ll} a_{ij} > 0 & \text{for all } i, j \in \{1, \dots, n\}\,. \\ 2a_{ii} > \sum_{j=1}^{n} a_{ij} & \text{for } i \in \{1, \dots, n\}\,. \end{array}$$

Show that A is invertible.

[**Hint.** For any real numbers b and c, $|b + c| \leq |b| + |c|$ and $|bc| = |b||c|$, where $|\cdot|$ denotes absolute value.]

TURN OVER

11. Answer all parts.

(a) Let $x_1, x_2, \ldots, x_n$ be a basis for $S \subseteq \mathbb{R}^m$.

i. If $n \leq m$, how might $\{x_1, x_2, \ldots, x_n\}$ be used to find a basis for $\mathbb{R}^m$?

ii. If $\alpha \neq 0$, show that the collection $\{\alpha x_1, \ldots, x_n\}$ also provides a basis for S.

iii. Prove that the collection $\{\alpha x_1 + x_2, x_1 + \beta x_2, x_3, \ldots, x_n\}$ provides a basis for S if and only if $\alpha\beta \neq 1$.

(b) The following system of 4 equations is for a 2-good economy.

$$Q_1^d = K_1 P_1^{a_{11}} P_2^{a_{12}} Y^{b_1}; \quad Q_1^s = M_1 P_1^{n_1}$$
$$Q_2^d = K_2 P_1^{a_{21}} P_2^{a_{22}} Y^{b_2}; \quad Q_2^s = M_2 P_2^{n_2}$$

Q_i^s denotes supply, Q_i^d denotes consumers' demand, P_i denotes price for good $i \in \{1, 2\}$. Y denotes consumer income while M_i and K_i are constants, $i \in \{1, 2\}$. Let $q_i^s \equiv \ln Q_i^s$, $q_i^d \equiv \ln Q_i^d$, $p_i = \ln P_i$, $m_i = \ln M_i$, and $k_i = \ln K_i$ for $i \in \{1, 2\}$ and $y = \ln Y$.

i. Rewrite the above system of equations using the logarithmic transformations and prove that a_{ii} is the own price elasticity of demand for good i, a_{ij} is the cross price elasiticity of demand of good i with respect to good j, b_i is the income elasticity of demand for good i, and n_i is the own price elasticity of supply for good i where $i, j \in \{1, 2\}$.

ii. Solve for equilibrium prices, i.e. prices at which supply equals demand for each good. State, but do not prove, any invertibility conditions needed for your answer.

iii. Suppose the government imposes a percentage tax t on the consumers of good 1. What is the price of good 1 faced by its consumers? Use this new price expression in the system of equations and solve for equilibrium prices. Again, state without proof any invertibility conditions needed.

iv. Assume $(a_{11} - n_1)(a_{22} - n_2) - a_{12}a_{21} > 0$ and $n_1 > 0$. How does an increase in the tax affect the price of good 1 and good 2? Explain your answer.

Subsection B.II: Answer <u>only one question</u> from this sub-section

12. Assume that we can model unemployment as

$$U_t = \alpha + \beta U_{t-1} \text{ with } \alpha > 0, \beta > 0;\ t = 1, 2, 3, ...$$

where U_t is the unemployment rate at time t.

   (a) Solve this difference equation given U_0. Solve for the natural rate of unemployment (steady state) and determine the necessary and sufficient condition for U_t to converge to the natural rate of unemployment.

   (b) Suppose that there are occasional shifts in the demand for labor causing it to sometimes rise and sometimes fall. These shifts translate into occasional decreases and increases in the unemployment rate. The modified model of unemployment that captures these shifts is

$$U_t = \alpha + \beta U_{t-1} + e_t$$

   where e_t denotes mean zero random shocks. Find an expression for U_t in terms of U_0 and the random shocks.

   (c) Based on (b), discuss whether the unemployment rate converges over time.

13. The inhabitants in town i, $i = 1, 2, ...n$, each derive consumer surplus $S_i = \frac{1}{2}(a_i - p_i)^2 + m_i$, where m_i is money income, when confronted with a price p_i for a good that they consume in amount q_i. Assume that each town has 1 consumer for simplicity and demand is given by $-dS_i/dp_i$.

   (a) A monopolist can produce the good at zero variable cost and initially sets the same price p in every town to maximize his profits. Every town has a positive demand at this price. Find the profit maximizing price and the sum total of consumer surplus.

   (b) Now suppose he can charge different prices in each town. Find the profit maximizing prices and the sum total of consumer surplus. What happens to the sum total of consumer surplus?

TURN OVER

Sub-section B.III: Answer only one question from this sub-section

14. Answer all parts.

(a) Consider two hypotheses H_0 and H_1 which are mutually exclusive and exhaustive. Suppose that we have observed some data y. Show that the posterior odds, $p(H_0|y)/p(H_1|y)$, are given by

$$\textit{posterior odds} = \textit{likelihood ratio} \times \textit{prior odds}$$

(b) A new HIV home test has been introduced. The test is used in a population with HIV prevalence of 1/1000. Let H_0 denote the hypothesis that the individual is truly HIV positive, and y is the observation that they test positive. H_1 denotes the hypothesis that the individual is truly HIV negative. The test is claimed to have the properties $p(y|H_0)$=0.95 and $p(y|H_1)$=0.02.

i. 100,000 individuals are tested. Given prior information on HIV prevalence (1/1000) we would expect 100 to test positive and 99,900 to test negative. Using the information provided calculate the expected status of those tested by finding values for cells [1]-[6] in Table 1.

ii. Find the posterior odds $\frac{p(H_0|y)}{p(H_1|y)}$

iii. Show that over 95% of those testing positive for HIV will not have the disease.

iv. How would your inference differ if prior beliefs are ignored? Explain your answer.

Table 1: Expected Status of 100,000 tested individuals

	HIV−	HIV+	Total
Negative Test	[1]	[2]	[3]
Positive Test	[4]	[5]	[6]
Total	99,900	100	100,000

15. Suppose you wish to estimate the expected length of unemployment spells in Britain. You believe that unemployment spells, X, are drawn from an exponential distribution, with density

$$f(X) = \theta \exp(-\theta X)$$

(a) Find the expected value $E(X)$ of X.

(b) After 8 weeks you collect data on a random sample of individuals. For those that are now employed the durations of unemployment are known. For those still unemployed, the durations are censored at 8 weeks.

i. Derive the maximum likelihood estimator θ^{ML} of θ. What is the estimated mean unemployment rate?

ii. Derive the variance of θ^{ML}.

END OF PAPER.