ECONOMICS TRIPOS PART IIA

Thursday 9 June 2011 9-12

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections: A and B.

Each Section carries 50% of the total marks.

*Section A is divided into three sub-sections and candidates are required to answer a total of **SIX** questions, with at least one question from each sub-section.*

*Section B is divided into three sub-sections and candidates are required to answer a total of **THREE** questions, with one question from each sub-section.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 2	Approved calculators allowed
Rough Work pads	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Section A: Answer six questions from this section

Sub-section A.I: Answer at least one question from this sub-section

1. Answer all parts.

   (a) Let A be an $n \times m$ matrix. Show that $tr(A^T A) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2$, where $tr(\cdot)$ denotes the trace of a matrix.

   (b) Let A and B be two matrices such that AB and BA are well defined. Show that $tr(AB) = tr(BA)$.

2. Answer all parts

   (a) Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\, 10x_1 + 3x_2 = 0\}$. Find a collection of linearly independent vectors in $\mathbb{R}^3$ whose span is S. Can any $x \in \mathbb{R}^3$ be expressed as a linear combination of the linearly independent vectors you found? Explain your answer.

   (b) Let $x = (1, 4, -2)^T$ and $y = (-2, -3, 7)^T$ and $z = (4, 1, z_3)^T$. For what value(s) of z_3 is z in the span of x and y?

3. Consider the following quadratic form for $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$.

   (a) Find the symmetric matrix A which represents this quadratic form, i.e. satisfies $Q(x) = x^T Ax$.

   (b) Determine whether $Q(\cdot)$ is positive semi definite. (*Hint.* 2 is an eigenvalue of A.)

Sub-section A.II: Answer <u>*at least one question*</u> *from this sub-section*

4. Answer all parts.

   (a) Show that $y \ln(2 + y^3)$ is an increasing function of y on $(0, \infty)$.

   (b) Let $F(x, y) = x + y \ln(x^3 + y^3 + 1)$, and consider the curve

   $$F(x, y) = 1 + \ln 3$$

   on the positive quadrant of the x-y plane (i.e., where $x \geq 0, y \geq 0$).

      i. Show that the only point (x, y) on this curve with $x = 1$ is the point $(1, 1)$.

      ii. Find the slope of the tangent line to this curve at $x = 1$.

5. Let $P(x) = x^4 + Ax^3 + Bx^2 + Cx - 3$, where A, B, C are constants. Suppose that the limits $\lim_{x\to 1} \frac{P(x)}{x^2-1}$ and $\lim_{x\to -1} \frac{P(x)}{x^2-1}$ exist, are finite, and that

   $$\lim_{x\to 1} \frac{P(x)}{x^2 - 1} = \lim_{x\to -1} \frac{P(x)}{x^2 - 1}.$$

   Evaluate A, B, C.

6. Suppose that the capital stock k of an economy evolves as

   $$\frac{dk(t)}{dt} = y(t) - \beta t$$

   where the production function is

   $$y(t) = \alpha t k(t)$$

   $y(\cdot)$ is output, t denotes time, and α and β are constants. Solve to find the expression for the economy's capital stock as a function of time t and initial capital stock k_0. What is the steady state capital stock?

TURN OVER

Sub-section A.III: Answer <u>*at least one question*</u> *from this sub-section*

7. Consider the equation $U = x_1^\gamma + x_2^\gamma$. An econometrician has estimated γ using n observations and concludes that γ is asymptotically normally distributed with mean $\bar{\gamma}$ and variance σ^2/n. Find the aymptotic mean and variance of U when $x_1 = 2$ and $x_2 = 3$? Explain your answer. (*Hint.* Consider a first-order Taylor expansion of U and recall that for a scalar a, $\frac{da^x}{dx} = a^x \ln a$.)

8. Consider the following data from 50 Bernoulli trials.

0	0	1	0	0	1	0	0	0	0
0	1	0	1	1	1	0	0	0	0
1	0	1	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	1	0	1	0	0

Let θ be the probability of observing a one for any single trial.

   (a) Find the likelihood of any sequence of s trials containing y ones.

   (b) If the prior on θ is uniform such that $p(\theta) = 1$, find an expression for the posterior distribution for θ after the first 10 trials above. What is the maximum likelihood estimator of θ?

   (c) Find an expression for the posterior after *another* 30 trails above. In what sense has the prior been updated?

9. Suppose X_n and Y_n are two estimators for θ, a real number. Suppose $\mathbb{E}X_n = \theta$, $Var\,(X_n) = cn^{-1}$, where c is a real positive constant, $\mathbb{E}Y_n = \theta + n^{-1}$, and $Var\,(Y_n) = n^{-1}$.

   (a) Compute the mean square error of X_n and of Y_n about θ.

   (b) Using the mean square error criterion, for what values of c will you choose X_n over Y_n?

Section B: Answer three questions from this section

Sub-section B.I: Answer only one question from this sub-section

10. Answer all parts

 (a) Consider the following system of equations,

$$\begin{aligned} -3x + 4y &= 8 \\ 6x + ty &= s \end{aligned}$$

 where t and s are scalars. Find values of t and s so that

 i. the system has a solution,

 ii. the system has no solution, and

 iii. the system has infinitely many solutions.

 Provide a geometric interpretation for each of the above cases.

 (b) A *linear transformation* is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$, where $n \geq 1, m \geq 1$, such that for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and scalar c, $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$.

 i. Show that $T(\sum_{k=1}^{K} c_k x^k) = \sum_{k=1}^{K} c_k T(x^k)$ for any scalars c_k and $x^k \in \mathbb{R}^n$, $k = 1, \ldots, K$ and that $T(0_{n\times 1}) = 0_{m\times 1}$, where $0_{n\times 1}$ denotes the zero vector in $\mathbb{R}^n$ and $0_{m\times 1}$ denotes the zero vector in $\mathbb{R}^m$.

 ii. Which of the mappings $f^1 : \mathbb{R}^3 \to \mathbb{R}^3$, $f^2 : \mathbb{R}^2 \to \mathbb{R}^2$, $f^3 : \mathbb{R}^2 \to \mathbb{R}$, and $f^4 : \mathbb{R}^3 \to \mathbb{R}^2$ given below are linear transformations? Explain your answers.

$$f^1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad f^2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 + 1 \\ x_1 - x_2 - 2 \end{pmatrix},$$

$$f^3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2, \quad f^4\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_1 - x_3 \end{pmatrix}$$

 iii. For each linear transformation f you identify above, provide an appropriate matrix A such that $f(x) = Ax$.

 iv. Let $T^1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T^2 : \mathbb{R}^m \to \mathbb{R}^p$, where $p \geq 1$, be linear transformations. Show that $T^3 : \mathbb{R}^n \to \mathbb{R}^p$ defined by $T^3(x) = T^2(T^1(x))$ for $x \in \mathbb{R}^n$ is a linear transformation.

TURN OVER

11. Answer all parts.

(a) Consider the following equation system where Y and r are endogeneous variables and b, G, I^0, a, m, M^0, h, and M^s are exogeneous parameters.

$$Y(1-b) - G = I^0 - ar$$
$$mY + M^0 - hr = M^s$$

i. Write the equation system in matrix form and provide conditions on the exogeneous parameters so that this system can be solved uniquely for Y and r.

ii. Solve for Y and r using Cramer's rule under these conditions.

iii. Determine the effect on Y and r of a change ΔG in G.

(b) Consider the following definition: a set $X \subseteq \mathbb{R}^n$, where $n \geq 1$, is *convex* if $\lambda x + (1-\lambda)y \in X$ whenever $x \in X$, $y \in X$, and $0 \leq \lambda \leq 1$.

i. Show that $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|x_i \geq 0, i = 1, \ldots, n\}$ is convex.

ii. Let $p \in \mathbb{R}^n$ with $p_i > 0, i = 1, \ldots, n$, and let $w > 0$ be a scalar. Show that $B(p, w)$ is convex, where

$$B(p,w) = \left\{x \in \mathbb{R}^n_+ \left|p^T x \leq w\right.\right\}.$$

iii. Suppose the function $u : \mathbb{R}^n_+ \to \mathbb{R}$ satisfies $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, whenever $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n_+$, and $0 \leq \lambda \leq 1$. Show that $D(p, w; u)$ is convex, where

$$D(p,w;u) = \{x \in B(p,w) \,|u(x) \geq u(y) \text{ for all } y \in B(p,w)\}.$$

(c) Suppose A is a $n \times n$ matrix which satisfies $AA = A$.

i. Show that A is non-singular if and only if $A = I$, where I is the identity matrix.

ii. Show that all eigenvalues of A are 0 or 1.

Subsection B.II: Answer <u>only one question</u> from this sub-section

12. A central planner controls an economy with two sectors, producing $x_1 \geq 0$ and $x_2 \geq 0$. This economy is small relative to the rest of the world and so takes as given the world prices $p_1 > 0$ and $p_2 > 0$ for the goods. Labour is the only input and is available in fixed total amount $L^0 > 0$. The production functions in the two sectors are

$$\begin{aligned} x_1 &= a_1 L_1^b \\ x_2 &= a_2 L_2^b \end{aligned}$$

where $L_1 \geq 0$ and $L_2 \geq 0$ denote labour input in sectors 1 and 2 respectively, $a_1 > 0$, $a_2 > 0$, and $0 < b < 1$. Suppose the planner wishes to maximize the *value* of national output at these world prices.

(a) State the planner's problem and find the optimum labour input for each sector. Explain your answer.

(b) What is the effect on the value of national output of a small change in labour supply? Explain your answer.

(c) What is the effect on the value of national output of a small change in the world price p_1? Explain your answer.

13. Suppose we order a continuum population according to increasing income, i.e. we represent the society with the interval $[0,1]$. For any fraction $r \in [0,1]$, let

$$g(r) = \frac{\text{total income of the bottom } r \text{ of the population}}{\text{total income of the whole society}}.$$

For example, $g(0.1)$ is the share of the bottom 10% of the society. Assume that g is differentiable with a continuous derivative.

(a) Show that g is a convex function.

(b) Suppose that the income distribution in this society is given by the continuous density function $f(w) = K/\sqrt{w}$ with support $(0,4]$, and suppose that F is the associated cumulative distribution so that

$$F(w) = \text{ fraction of the population with income less than } w$$

i. Evaluate K.

ii. Find the median income in this society.

iii. Find the average income.

(c) The **Gini index** G of this society is defined as the area between the graphs of the line $y = x$ and the curve $y = g(x)$, i.e. $G = \int_0^1 (x - g(x))dx$. Evaluate the Gini index of this society.

TURN OVER

Sub-section B.III: Answer only one question from this sub-section

14. Let $X_1, X_2, ..., X_n$ denote a sequence of i.i.d. random variables with finite mean $E(X)$ and variance σ_X^2. Let

$$S_n = \frac{\sum_{i=1}^{n} X_i}{n}.$$

(a) Show that for any $\varepsilon > 0$

$$\lim_{n \to \infty} \mathrm{P}(|S_n - E(X)| > \varepsilon) = 0,$$

i.e. S_n converges in probability to $E(X)$.

(b) Suppose $\varepsilon > 0$ and answer the following.

i. Given $\delta > 0$ find n such that

$$\mathrm{P}(|S_n - E(X)| \leq \varepsilon) \geq 1 - \delta.$$

ii. For $\varepsilon = 0.1\sigma_X$ and $\delta = 0.001$, find n so that S_n is within $\pm 0.1\sigma_X$ of $E(X)$ with probability at least 0.999.

(c) Using a suitable normalisation of $S_n - E(X)$ and the Central Limit Theorem, find an approximation to $\mathrm{P}(|S_n - E(X)| \leq \varepsilon)$, where $\varepsilon > 0$.

(d) For $X_i \sim \mathit{Bernoulli}(p)$, where $E(X) = p$, find n large enough such that

$$\mathrm{P}(|S_n - p| \leq 0.1) \geq 0.95.$$

15. A random variable y has the probability density function

$$f(y) = \begin{cases} (1-\theta) + 2\theta y & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

with $-1 < \theta < 1$. There are n observations y_i, $i = 1, ..., n$, drawn independently from this distribution.

(a) Derive the cumulative distribution function of y and the expected value of y.

(b) Write the expected value of y as a function of θ and use this to suggest an estimator for θ based on the sample mean $\overline{y}$.

(c) Write the log-likelihood function for θ and derive the first-order condition for the maximum likelihood estimator of θ.

END OF PAPER.