UNIVERSITY OF CAMBRIDGE

ECONOMICS TRIPOS PART IIA

Thursday 7 June 2012 9:00-12:00

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections - A and B.

Section A is divided into three sub-sections and candidates are required to answer a total of **SIX** questions, **with at least one** question from each sub-section.

Section B is divided into three sub-sections and candidates are required to answer a total of **THREE** questions, **with one** question from each sub-section.

Each section carries 50% of the total marks. Within each section, each question carries equal weight.

Write your **candidate number** not your name on the cover of each booklet.

Write legibly.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 1	Approved calculators allowed
Rough work pads	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A: Answer SIX questions from this section.

Sub-section A.I: Answer at least one question from this sub-section.

1 Consider an $m \times n$ matrix A, where $m \neq n$. Which of the subspaces $RowA$, $ColA$, $NulA$, $RowA^T$, $ColA^T$, and $NulA^T$ are in $\mathbb{R}^m$ and which in $\mathbb{R}^n$? Which of these subspaces are distinct subspaces?

(a) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix such that $AB = 0$. Explain why $ColB$ is a subspace of $NulA$, and show that

$$rankA + rankB \leq n.$$

2 Let $\mathbf{e}_n$ be the $n \times 1$ vector with ones in all its entries, i.e.

$$\mathbf{e}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

and let $J = \frac{1}{n}\mathbf{e}_n\mathbf{e}_n^T$.

(a) Show that 1 is an eigenvalue of J and find a corresponding eigenvector.

(b) Show that 0 is an eigenvalue of J.

3 (a) Show that if U and V are orthogonal matrices, then UV is orthogonal.

(b) Consider the quadratic form

$$Q(x, y) = ax^2 + 2bxy + dy^2.$$

Write the matrix A that corresponds to this quadratic form, and verify that A is positive definite if $ad > b^2$ and $a > 0$.

Sub-section A.II: Answer at least one question from this sub-section.

4 Data are collected on N independent trials which consist of flipping a coin. For trial $i\,(i = 1, ..., N)$, $y_i \in \{0, 1\}$ records 1 for a head and 0 a tail. We observe $x = \sum_{i=1}^{N} y_i$ heads and $N - x$ tails.

(a) Consider the following mutually exclusive hypotheses:

H_0: the coin is fair

H_1: the probability of a head is 1/4.

Using Bayes' Theorem, find an expression for the relative probability of H_0 and H_1, *given* the data $\left(\sum_{i=1}^{N} y_i\right)$ and prior beliefs over the two hypotheses.

(b) If the prior probabilities are, respectively, $\Pr(H_0) = 1/2$ and $\Pr(H_1) = 1/2$, find an expression for the posterior odds given 2 heads in 3 trials.

5 (a) State the Neyman-Pearson lemma.

(b) The pdf of the Poisson distribution with mean λ is

$$f(x) = \lambda^x e^{-\lambda}/x! \qquad x = 0, 1, 2, \ldots,$$

and the sum of a random sample of n observations from a Poisson distribution is Poisson distributed with mean $n\lambda$.

Show that, for a random sample from a Poisson distribution, the critical region $\sum_{i=1}^{n} x_i \leq 2$ is uniformly most powerful (UMP) for testing $H_0 : \lambda = \frac{1}{2}$ against $H_1 : \lambda < \frac{1}{2}$ at a particular significance level.

(c) Write down a formula for computing the significance level when $n = 12$.

6 The pdf of an exponential distribution is

$$f(x; \theta) = \theta^{-1} \exp(-x/\theta), \qquad 0 \leq x \leq \infty,$$

where θ is the mean. Find the maximum likelihood estimator of θ based on a random sample of n observations, write down its asymptotic distribution, and hence obtain a large sample 95% confidence interval for θ.

(TURN OVER)

Sub-section A.III: Answer at least one question from this sub-section.

7 Let the function $f : (0, \infty) \to \mathbb{R}$ be defined as $f(x) = x^{\ln x}$.

(a) Evaluate $\lim_{x \to 0^+} f(x)$.

(b) Show that f is convex.

8 A monopolist produces Q units of a good, at total cost $c(Q)$. The price is then $P(Q)$, where P is the inverse demand function. Assume $P' < 0$, $P'' \leq 0$ and $c' \geq 0$.

(a) What is the profit function? What is the First Order Condition (FOC) for an interior optimum $Q^* > 0$? Is a point satisfying the FOC necessarily an optimum?

(b) Assuming that the FOC is satisfied, state, with justification, whether the following changes increase or decrease the optimal Q^*:

(i) It becomes more expensive to produce each unit of the good: $c'(Q)$ is increased for every Q.

(ii) The demand function shifts up: the whole function $P(Q)$ is increased by a constant.

9 A consumer has utility $\ln x + m$, where x is the amount of the good consumed, and m is money. He has wealth $w > 0$, and purchases the good at price p, and is subject to a budget constraint. So he maximizes $\ln x + m$ subject to $xp + m = w$ and $m \geq 0$.

(a) Write down the maximisation problem and the associated Lagrangian (either a standard or a Kuhn-Tucker Lagrangian).

(b) Solve the problem.

(c) If $v(p, w)$ is the indirect utility function, evaluate $\dfrac{\partial v}{\partial w}$.

SECTION B: Answer THREE questions from this section.

Sub-Section B.I: Answer only one question from this sub-section.

10 (a) State Cramer's rule.

(b) Consider the IS-LM model, summarised as follows. Let Y represent national income, C spending by consumers, I investment spending by firms, T taxes, and G government spending. Let $0 < a < 1$ be the marginal propensity to consume. Assume that the consumption function is given by

$$C(Y, r) = a\,(Y - T) - cr,$$

where $Y - T$ is disposable income, r is the rate of interest, and $c > 0$. Assume that the investment function is given by

$$I = I_0 - br,$$

where $b > 0$. Next, let the demand for real money balances be

$$L = \mu Y + M_0 - \lambda r,$$

with $\mu, \lambda > 0$, and assume that the real money supply M_s is exogenous and fixed.

(i) Derive the IS and LM curves by looking for the equilibrium in the goods and money market.

(ii) Using Cramer's rule, find the equilibrium income and interest rate in the IS-LM model by solving the corresponding system of linear equations.

(iii) Suppose there is an increase in government spending $\Delta G > 0$ and a simultaneous increase in taxes $\Delta T > 0$. What condition needs to be satisfied in order to ensure that the equilibrium income does not decrease after these changes?

11 (a) Let

$$A_k = \begin{bmatrix} A & 0 \\ 0 & I_k \end{bmatrix} \text{ and } D_k = \begin{bmatrix} I_k & 0 \\ C_k & D \end{bmatrix}$$

where A, D are $n \times n$ matrices, I_k is a $k \times k$ identity matrix, C_k is a $n \times k$ matrix and 0 are conformable matrices of zeros. Show by induction that

$$\det A_k = \det A$$

and

$$\det D_k = \det D$$

for all $k = 1, \ldots, n$.

(b) Show that if A, C, and D are $n \times n$ matrices,then

$$\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det A \det D.$$

Sub-Section B.II: Answer only one question from this sub-section.

12 (a) Let X be a continuous random variable and $g(\cdot)$ a non-negative function with domain the real line. Prove that

$$\Pr[g(X) \geq k] \leq \frac{E[g(X)]}{k} \qquad (\forall k > 0).$$

(b) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed with mean μ and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Prove that $\bar{X}_n$ *converges in probability* to μ, i.e., show that for every $\epsilon > 0$ $\lim_{n\to\infty} \Pr(|\bar{X}_n - \mu| < \epsilon) = 1$.

13 A normal distribution with mean μ and variance σ^2 is denoted $N(\mu, \sigma^2)$, and its pdf is given by

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}.$$

Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean zero and variance σ^2, that is $N(0, \sigma^2)$.

(a) Find $\widetilde{\sigma}^2$, the ML estimator of σ^2. Write down its (exact) distribution and hence determine whether it is the minimum variance unbiased estimator (MVUE).

(b) Find the asymptotic variance of $\widetilde{\sigma}$, the ML estimator of σ.

(c) Given that $E|X| = \sigma\sqrt{2/\pi}$ for a normal distribution, show that

$$\widehat{\sigma} = \sqrt{\frac{\pi}{2}} \frac{\sum_{i=1}^{n} |X_i|}{n}$$

is an unbiased estimator of σ. Find its variance. What is its asymptotic efficiency?

Sub-Section B.III: Answer only one question from this sub-section.

14 (a) Define the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ as

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

A series $U(x) = \sum_{n=0}^{\infty} u_n(x)$ admits term-by-term differentiation if $U'(x) = \sum_{n=0}^{\infty} u_n'(x)$.

Assuming that $\exp x$ admits term-by-term differentiation, show that $(\exp x)' = \exp x$.

(b) Using definitions of differentiation and integration, prove the *Fundamental Theorem of Calculus*: If f is continuous on $[a, b]$, and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.

(c) The natural logarithm function is defined as:

$$\text{for } x > 0, \text{ let } \ln x = \int_1^x \frac{1}{t} dt.$$

Prove that $\ln x$ and $\exp x$ are the inverse functions of each other.

15 In an auction for a single object, two bidders have valuations v and w, respectively, drawn independently from a uniform distribution on $[0, 1]$. Both of them use the same strictly increasing bidding function $b(\cdot)$; that is, if a bidder has valuation x, she submits a bid of $b(x)$. The highest bidder wins the object and pays her bid.
If bidder 1 with valuation v bids B and wins, she gets a payoff of $v - B$. Knowing that her opponent is bidding according to the bidding function $b(\cdot)$, she will win with probability $b^{-1}(B)$, i.e. the probability that the other player has bid less than B.

So if bidder 1 has value v and bids B, she gets utility:

$$u(v, B) = (v - B)\, b^{-1}(B)$$

(a) If bidder 1 has value v, what is the first order condition for a bid B to maximize utility?

(b) Since utility is maximized when bidder 1 takes the optimal strategy $B = b(v)$, show that $b(v)$ satisfies the differential equation:

$$b'(v) = 1 - \frac{1}{v} b(v)$$

(c) Recall that the differential equation $\frac{dy}{dt} = p(t)\,y + q(t)$ has solution:

$$y = \left[\int q(t) e^{-\int p(t)dt} dt \right] e^{\int p(t)dt}.$$

Find $b(v)$, using the initial condition $b(0) = 0$.

END OF PAPER