UNIVERSITY OF CAMBRIDGE

ECONOMICS TRIPOS PART IIA

Thursday 6 June 2013 9:00-12:00

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections - A and B.

Section A is divided into three sub-sections and candidates are required to answer a total of **SIX** questions, **with at least one** question from each sub-section.

Section B is divided into three sub-sections and candidates are required to answer a total of **THREE** questions, **with one** question from each sub-section.

Each question within each section will carry equal weight.

Write your **candidate number** not your name on the cover of each booklet.

Write legibly.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
20 Page booklet x 1	Approved calculators allowed
Rough work pads	
Tags	

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

Sub-section A.I: Answer at least one question from this sub-section

1 Let

$$A = \begin{pmatrix} 5 & 0 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 2 & -1 & 3 & 0 & 0 \\ 4 & -2 & -2 & c+2 & 0 \\ 0 & 1 & 0 & -1 & c \end{pmatrix}.$$

Find the value for c such that the numbers 3 and 5 are eigenvalues of A. Find an eigenvector for each of these two eigenvalues.

2 Let A be a $(n+m) \times (n+m)$ matrix that can be written as

$$A = \begin{pmatrix} I_n & \mathbf{0} \\ C & I_m \end{pmatrix}$$

where I_n and I_m are identity matrices of dimensions $n \times n$ and $m \times m$ respectively, $\mathbf{0}$ is an $n \times m$ matrix of zeros, and C is an $m \times n$ matrix. Show that A is invertible and find its inverse.

3 Are the following sets of vectors linearly dependent or independent? Justify your answers.

(a) $$S_1 = \left\{ \begin{pmatrix} 3 \\ 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 8 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 5 \end{pmatrix} \right\}$$

(b) $$S_2 = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \\ -1 \end{pmatrix} \right\}$$

(c) $$S_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.5 \\ -11 \end{pmatrix} \right\}$$

(d) $$S_4 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Sub-section A.II: Answer at least one question from this sub-section

4 Given n observations from a distribution with density function

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & , \quad 0 < x < 1, \qquad 0 < \theta < \infty \\ 0 & , \quad \text{elsewhere.} \end{cases},$$

find the maximum likelihood estimator of θ. What is the asymptotic distribution of this estimator ?

Find the mean of the above distribution and hence construct a simple method of moments estimator of θ.

5 In the general linear model,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

$\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X}$ is an $n \times k$ full-rank matrix of nonstochastic explanatory variables, $\boldsymbol{\beta}$ is a $k \times 1$ vector of regression coefficients and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of disturbances with $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \sigma^2\mathbf{I}$.

(a) Show that the OLS estimator, $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, is unbiased and derive its variance-covariance matrix. What assumptions, if any, are needed for $\mathbf{b}$ to be consistent?

(b) If the explanatory variables include a constant, show that the OLS residuals will sum to zero.

6 X and Y are independent discrete random variables taking values in $\{0, 1, 2, ..., n\}$ and binomial distribution given by

$$p_X(j) = p_Y(j) = \binom{n}{j} p^j q^{n-j}, \tag{1}$$

where $p, q > 0$ and $p + q = 1$.

(a) Find the moment generating function of $Z = X + Y$, and use this to determine the distribution of Z.

(b) Consider the following linear regression model

$$\ln y = \alpha + x\beta + \varepsilon.$$

x is a scalar regressor and α and β are unknown parameters. You are also told that $\varepsilon \sim i.i.d(0, \sigma^2)$.

Show that $e^{\widehat{E[\ln(y)]}}$ is a biased estimator of $E(y|x)$.

Sub-section A.III: Answer at least one question from this sub-section

7 Let the function $f : (-1, \infty)$ be defined as $f(x) = \ln(1 + x)$.

(a) Is f convex, concave, or neither? Justify your answer.

(b) Evaluate the fifth and the sixth derivatives of f at the point $x = 0$.

(c) Find the Taylor series expansion of f around the point $x = 0$.

8 Given two differentiable functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$,

(a) State the integration by parts formula and give a brief proof.
(b) Evaluate the integral

$$\int_0^1 x^2 e^x dx.$$

9 A temperate consumer maximizes $u(f, d) = -(f - 10)^2 - (d - 10)^2$ over food f and drink d subject to a budget constraint $f + d \leq w$. (There are no non-negativity constraints.)

a) Solve the problem using Lagrangian methods.

b) Suppose the price of drink rises from 1 to 1.1, so the budget set changes to $f + 1.1d \leq w$. Find an approximation for the change in utility.

SECTION B

Section B.I: Answer only one question from this sub-section

10 Consider an economy with three sectors, agriculture (A), manufacturing (M) and tourism (T), with input-output matrix

	A	**M**	**T**
A	0.2	0.1	0.2
M	0.2	0.2	0.2
T	0	0	0.1

and demand vector

$$\mathbf{d} = \begin{pmatrix} 10 \\ 10 \\ 5 \end{pmatrix}.$$

(a) Suppose that the forecast of a tropical cyclone causes a decrease in the demand for tourism by two units. By how much should the production sectors adjust their production?

(b) Now suppose that, apart from the reduction in the demand for tourism, the anticipated cyclone generates an increase in the demand for manufacturing. By how much should demand for manufacturing increase, so that the overall effect on agriculture production is zero? Can the increase in the demand for manufacturing offset the negative effect of the cyclone on tourism?

Explain and justify your answers carefully.

11 Let A be a $n \times n$ matrix and E be the $n \times n$ elementary row replacement matrix, that multiplies the first row of a matrix by μ and adds it to the last row:

$$E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu & 0 & \cdots & 1 \end{pmatrix}$$

Prove that $\det(EA) = \det E \det A$ for all $n \geq 2$:

Section B.II: Answer only one question from this sub-section

12 A random sample, $X_1, ..., X_n$, is drawn from a normal distribution with mean μ and variance σ^2.

(a) When μ is known, show that

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

is an unbiased estimator of σ^2. What is its (exact) distribution and variance?

(b) When μ is unknown, show that the maximum likelihood estimator of σ^2 is biased. Hence obtain an unbiased estimator, denoted S^2.

(c) The variance of S^2 can be shown to be $2\sigma^4/(n-1)$. Compare this with the variance of $\widetilde{\sigma}^2$. Obtain an expression for the difference in the mean square errors (MSEs) of $\widetilde{\sigma}^2$ and S^2 and comment.

(d) Find the information matrix for μ and σ^2, and hence determine whether the sample mean, $\bar{X}$, and S^2 attain the Cramer-Rao lower bound.

NB The pdf for a normal distribution is

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp(\frac{-(y-\mu)^2}{2\sigma^2}), \qquad -\infty < y < \infty$$

13 It is conjectured that the high cancer rate among the ex- and current employees of a school could be due to nearby high voltage transmission lines. Eight cases of invasive cancer have been observed among 145 women staff members whose average age was between 40 and 44.

(a) Employees developed (or not) cancer independently of each other and the chance of cancer, θ, is the same for each employee. For any integer x between 0 and 145, find an expression for $\Pr(X = x|\theta)$, where X denotes the number of cancers among the 145 employees.

(b) The national incidence of this cancer is 3% in women aged 40-45. In analysing this data a *classical* hypothesis test, using a significance level of $\alpha = 0.10$, is formulated as

$$H_0 : \theta = 0.03 \quad \text{vs } H_1 : \theta > 0.03.$$

In this context, provide an interpretation of the quantity

$$\zeta = \sum_{i=8}^{145} \Pr(X = i|\theta = 0.03).$$

(c) Since θ is unknown we can compute the probability of observing the data according to a number of theories on θ. Theory A: $\theta = 0.03$ is based on the national cancer rate. Alternative theories B: $\theta = 0.04$, C: $\theta = 0.05$, and D: $\theta = 0.06$ were also postulated. The prior probabilities for each theory to be correct are $\Pr(A) = \frac{1}{2}$ and $\Pr(B) = \Pr(C) = \Pr(D) = \frac{1}{6}$.

The likelihood of observing the data given each theory, namely $\Pr(X = 8|j)$, $j = A, B, C, D$, is, respectively, 0.036, 0.096, 0.134, and 0.136.

(i) Find $\Pr(j|X = 8)$ for $j = A, B, C$ and D.

Comment on your results.

(ii) Using the results from part (i) compute $\Pr(\theta > 0.03)$. Compare your findings with the result from part (b) given that $\zeta = 0.07$.

Section B.III: Answer only one question from this sub-section

14 Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) Define what it means to say f is differentiable at a.

(b) Suppose that f is differentiable everywhere on $\mathbb{R}$, and attains its minimum value at a. Can we conclude that $f'(a) = 0$? If yes, prove it. If no, give a counter-example (that is, an example of a function differentiable everywhere, attaining its minimum at some point, but with non-zero derivative at that point).

(c) Suppose that g is differentiable everywhere on $\mathbb{R}$, and $g'(a) = 0$. Can we conclude that a is where g attains a local maximum or a local minimum? If yes, prove it. If no, give a counter-example.

(d) Suppose that f and g are two differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. If both of them attain their maximum values at $x = 0$, can we conclude that their product $f \cdot g$ attains its maximum value at $x = 0$? If yes, prove it. If no, give a counter-example.

(e) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function which is differentiable everywhere. Prove that either $f' \leq 0$ everywhere or $f' \geq 0$ everywhere.

15 A group of children plays rounds of rocks, paper and scissors. In the nth round fractions r_n, p_n and s_n play each move, where $r_n + p_n + s_n = 1$. Then in the subsequent round they adjust their strategies according to:

$$\begin{aligned} r_{n+1} &= (1-\epsilon)\, r_n + \epsilon s_n \\ p_{n+1} &= (1-\epsilon)\, p_n + \epsilon r_n \\ s_{n+1} &= (1-\epsilon)\, s_n + \epsilon p_n, \end{aligned}$$

where $\epsilon \in (0, 1)$ is a constant.

(a) Express this process as a difference equation in $v_n = \begin{pmatrix} r_n \\ p_n \end{pmatrix}$ only.

(b) Find the unique equilibrium v^*, and show that $v_n \to v^*$.

END OF PAPER