UNIVERSITY OF CAMBRIDGE

ECT2
ECONOMICS TRIPOS PART IIA

Thursday 5 June 2014 9:00-12:00

Paper 6

MATHEMATICS FOR ECONOMISTS AND STATISTICIANS

This paper consists of two Sections - A and B. Each Section carries equal weight.

Section A is divided into three sub-sections and candidates are required to answer a total of **SIX** questions, **with at least one** question from each sub-section.

Section B is divided into three sub-sections and candidates are required to answer a total of **THREE** questions, **with one** question from each sub-section.

Write your **candidate number** (not your name) on the cover of each booklet.

Write legibly.

STATIONERY REQUIREMENTS
20 Page booklet x 1
Rough work pads
Tags

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAMINATION
Calculator - students are permitted to bring an approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Sub-section A.I: Answer at least one question from this sub-section

1 Let A and B be square matrices of the same dimension. Are the following statements true or false? Justify your answers.

(a) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det A}$.

(b) $\det(A+B) = \det(A) + \det(B)$.

(c) If A is idempotent ($A^2 = A$) then $\det(A) = 1$.

2 (a) Let U be a set of vectors in $\mathbb{R}^3$ whose second and third entries are equal. Every vector in U has a unique representation

$$\begin{bmatrix} a \\ b \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (b-a) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

for any $a, b \in \mathbb{R}$. Is the set

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

a basis for U? Explain your answer.

(b) Write down a basis for the space of quadratic functions $f(x) = ax^2+bx+c$ satisfying $f(0) = 0$.

3 (a) Give the definition of an orthogonal matrix.

(b) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \subset \mathbb{R}^4$ and

$$\begin{aligned} A &= [\mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_1 \quad \mathbf{u}_4], \\ B &= [\mathbf{u}_1 \quad \mathbf{u}_4 \quad \mathbf{u}_2 \quad \mathbf{u}_3]. \end{aligned}$$

Show that A is orthogonal if and only if B is orthogonal.

(c) Show that if M is orthogonal then M^2 is orthogonal.

Sub-section A.II: Answer at least one question from this sub-section

4 If X is a random variable, then let $M_X(t)$ be the value of the moment-generating function of X at t.

(a) If $X_1, X_2, ...,$ and X_n are independent random variables and

$$Y = X_1 + X_2 + ... + X_n$$

show that

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

(b) A random variable X with Poisson distribution with parameter λ has probability mass function:

$$P(X = x) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, ...$$

Its moment-generating function is:

$$M_X(t) = e^{\lambda(e^t - 1)}$$

Find the probability distribution of the sum of n independent random variables X_1, $X_2, ..., X_n$ having Poisson distributions with the respective parameters $\lambda_1, \lambda_2, ..., \lambda_n$.

5 (a) Let $p(D|M_k) = \int p(D|M_k, \theta_k)p(\theta_k)d\theta_k$ denote the likelihood of model M_k integrated over the distribution of θ_k. Assuming that the set of models is given by $\Omega = \{M_0, M_1\}$, write down an expression for the posterior odds for M_1 against M_0.

(b) Let X denote a binomial random variable with parameters n and Ξ. The prior distribution of Ξ is a beta distribution with parameters τ and ν. Show that the posterior distribution of Ξ given $X = x$ is a beta distribution with parameters $x + \tau$ and $n - x + \nu$.

Definition: A random variable R has distribution $Beta(\tau, \nu)$ if its pdf has the form

$$f_R(r) \propto \begin{cases} r^{\tau-1}(1-r)^{\nu-1} & 0 \le r \le 1 \\ 0 & otherwise \end{cases}$$

6 (a) State Chebyshev's inequality.

(b) Let X_i $(i = 1, ..., n)$ denote a random sample from a Bernoulli distribution in which the probability that $X_i = 1$ is π. Find the mean and variance of the sample mean, $\overline{X} = \sum_1^n X_i/n$, and show that $\overline{X}$ converges in probability to π.

Sub-section A.III: Answer at least one question from this sub-section

7 Let $K \subset \mathbb{R}$ and $f : K \to K$ be a function. If $f(k) = k$, then k is called a **fixed point** of f.

(a) Prove that if K is a closed interval, and f is continuous, then f must have a fixed point.

(b) Given an open interval I, show by an example that a continuous function $f : I \to I$ does not necessarily have a fixed point.

8 (a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be quasiconcave.

(b) "If $u(x)$ and $v(x)$ are quasiconcave functions, $u(x) + v(x)$ is quasiconcave." By sketching a counterexample, or otherwise, show that this statement is not true. (Hint: A condition for a function $f : \mathbb{R} \to \mathbb{R}$ of a single variable to be quasiconcave is that it is single peaked; i.e. $f(x)$ is weakly increasing on $x \leq \alpha$ and weakly decreasing on $x \geq \alpha$ for some $\alpha \in \mathbb{R} \cup \{-\infty, \infty\}$.)

(c) Is $e^{\sqrt{x}+\sqrt{y}}$ quasiconcave? Explain your answer.

9 A firm maximizes profit $\pi = (1-w)h\,p$, where w is the wage offered to workers for each unit of time worked, and h is the amount of time they have to work each day, and $p = w - \frac{h}{2}$ is the number of workers the firm can hire under these terms. The firm is subject to a minimum wage requirement $\bar{w} \leq w$, where $\bar{w} = 1$.

(a) What are the Lagrangian first order conditions for this problem?

(b) Find the optimal choices (w, h).

(c) Find $\frac{d\pi^*}{d\bar{w}}$ at $\bar{w} = 1$, where $\pi^*(\bar{w})$ is the profit resulting from a minimum wage $\bar{w}$.

SECTION B

Sub-section B.I: Answer only one question from this sub-section

10 Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 3 \end{bmatrix}$$

(a) Find bases of
  (i) the row space of A,
  (ii) the column space of A,
  (iii) the null space of A.

(b) Comment on the dimensions of the spaces in (a).

(c) Find a basis of the eigenspace of: the smallest eigenvalue of A, giving an approximate numerical answer.

11 Suppose there is a fixed population and everyone belongs to the labour force. At a given point in time, each worker can be in one of three states:

- *employed under probation, E*
- *employed with tenure*, T (permanent employment until retirement), or
- *unemployed, U*.

Suppose that:

- Workers that are employed under probation become unemployed at a rate $0 < s < 1$ or employed with tenure at a rate $0 < \tau < 1$.
- Workers that are employed with tenure may lose their job and become unemployed at a rate $0 \leq \sigma < 1$ (redundancy in this case may be due to exogenous reasons and not because of their performance or productivity, e.g. because the firm they worked for closes down), but they never revert to employment under probation.
- Unemployed workers become employed under probation at a rate $0 < f < 1$ and employed with tenure at a rate $0 < \phi < f$.

(a) Express the dynamics of the different employment states as a system of difference equations in matrix form and explain why the matrix M that describes the dynamics is Markov.

(b) Let $s = 0.01,\ \tau = 0.1,\ \sigma = 0.05,\ f = 0.2,\ \phi = 0.01$. Show that the Markov matrix M is regular. What are the steady state rates of employment under probation, employment with tenure and unemployment?

(c) Suppose now that $\sigma = 0$, i.e. once a worker gets tenure, he can never become unemployed. What will happen to the rates of employment under probation, employment with tenure and unemployment in the long run?

Sub-section B.II: Answer only one question from this sub-section

12 Let $X_1, X_2, ..., X_n$ denote n *i.i.d.* random variables. Let $\widehat{\theta}$ be an estimator of an unknown parameter θ.

(a) Show that the mean square error (MSE) of an estimator $\widehat{\theta}$, $E[(\widehat{\theta}-\theta)^2]$, is given by $\text{Var}(\widehat{\theta}) + [E(\widehat{\theta}) - \theta]^2$.

(b) Suppose $X_1, X_2, ..., X_n$ come from a normal distribution, and consider the following two estimators of the variance (σ^2), where $\bar{X}$ is the sample mean of X_i:

$$\begin{aligned} \widehat{\sigma}_n^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \\ S_n^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{aligned}$$

Determine: $E[\widehat{\sigma}_n^2]$, $E[S_n^2]$, $\text{Var}(\widehat{\sigma}_n^2)$, and $\text{Var}(S_n^2)$.

Which estimator is preferable? Justify your answer.

Hint: $\frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2$

(c) Consider the following model for X_i with probability mass function $f(x;\theta)$:

$$\begin{aligned} X_i &= \theta + \varepsilon_i \\ f(x;\theta) &= \theta^x(1-\theta)^{1-x}, \quad 0 \le \theta \le 1, x = 0, 1 \end{aligned}$$

For each of the following estimators of θ

A. $\widehat{\theta}_2 = X_1 + X_2$

B. $\widehat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

C. $\widehat{\theta}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n} X_i$

(i) compute the sampling distribution of the estimators, and

(ii) compute the Mean Square Error.

Which estimators are consistent? Which estimator is preferred? Explain your answer.

13 A random variable Y with distribution $gamma(\theta, \gamma)$ has probability density function

$$f(y; \theta, \gamma) = \theta^{-\gamma} y^{\gamma-1} e^{-y/\theta} / \Gamma(\gamma), \qquad 0 \leq y < \infty, \qquad \theta, \gamma > 0,$$

where $\Gamma(\gamma)$ is the gamma function. The mean of Y is $\gamma\theta$, while the variance is $\gamma\theta^2$. Suppose that γ is known and that a random sample of size n is taken.

(a) Find the maximum likelihood (ML) estimator of the parameter θ and determine whether it is the minimum variance unbiased estimator (MVUE).

(b) Show that the sum of the sample observations is a sufficient statistic for θ.

(c) Now suppose that $\theta_i, i = 1, ..., n$, is different for each observation, with $\theta_i = \exp(\beta x_i)$, where $x_i\ , i = 1, ..., n$, is an (observed) explanatory variable and β is an unknown parameter. Write down the log-likelihood function. Let $\widetilde{\beta}$ denote the ML estimator of β. If $p\lim \left(\sum_{i=1}^{n} x_i^2 \right) / n = q$, where q is a positive constant, show that the variance of the limiting distribution of $\sqrt{n}(\widetilde{\beta} - \beta)$ is $1/\gamma q$.

Sub-section B.III: Answer only one question from this sub-section

14 Consider the graph of the function $f(x) = 1/x$ with domain $(1, \infty)$ and co-domain $\mathbb{R}$ so that the domain is represented on the horizontal x-axis, and the co-domain is represented on the vertical y-axis.

Now, consider the three dimensional object you would get if you rotate the graph around the x-axis, a "vase of infinite depth, lying on its side". (The top of the vase is a circle of radius 1, and the vase gets narrower as we move deeper.)

(a) Is the area between the graph and the x-axis finite? If so, compute it. If not, explain why not.

(b) Does the vase have finite volume? If so, compute it. If not, explain why not.

(c) The surface area of the vase is given by $\int_1^\infty f(x)\sqrt{1 + (f'(x))^2}dx$. Does the vase have finite surface area? If so, compute it; if not, explain why not.

(d) Which would take more paint: to fill the vase with paint, or to paint the surface of the vase? Give an explanation.

15 (a) Suppose the population $P(t)$ of the world evolves according to $\frac{dP}{dt} = \bar{K} - P$, where $\bar{K} > 1$ is a constant.

(i) The initial population is $P(0)$. Find $P(t)$.

(ii) What happens as $t \to \infty$?

(b) Suppose the population $P(t)$ and technology $K(t)$ of the world evolve according to $\frac{dP}{dt} = f(P, K) = K - P$ and $\frac{dK}{dt} = g(P, K) = P\left(\bar{K} - K\right)$, where $\bar{K} > 1$ is a constant.

(i) Find the stationary points and find which of these are stable.

(ii) Sketch the phase diagram showing the vectors $\left(\frac{dP}{dt}, \frac{dK}{dt}\right)$ as a function of (P, K) on the space $P \geq 0$, $K \geq 0$.

END OF PAPER