UNIVERSITY OF CAMBRIDGE

ECT2
ECONOMICS TRIPOS PART IIA

Thursday 4 June 2015 9:00am-12:00pm

Paper 6

MATHEMATICS & STATISTICS FOR ECONOMISTS

This paper consists of two Sections - A and B. Each Section carries equal weight.

Section A is divided into three sub-sections and candidates are required to answer a total of **SIX** questions, **with at least one** question from each sub-section.

Section B is divided into three sub-sections and candidates are required to answer a total of **THREE** questions, **with one** question from each sub-section.

Write your **candidate number** (not your name) on the cover of each booklet.

Write legibly.

STATIONERY REQUIREMENTS
20 Page booklet x 1
Rough work pads
Tags

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAMINATION
Calculator - students are permitted to bring an approved calculator

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

SECTION A

Sub-section A.I: Answer at least one question from this sub-section

1. Answer all parts.

   (a) Find an $a \in \mathbb{R}$ such that the following matrix A is singular:

$$A = \begin{pmatrix} a & a & -1 \\ -2 & 2 & 0 \\ 7 & 3 & -5 \end{pmatrix}$$

   (b) Are the following sets of vectors linearly independent?

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right\}, \quad S_2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \end{pmatrix} \right\}$$

   (c) What is the rank of the following matrix?

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 5 & -1 & -5 \end{pmatrix}$$

2. Which of the following are vector spaces? Justify your answers carefully.

   (a) $V_1 = \mathbb{Z}$, i.e. all integers, with standard addition and scalar multiplication.

   (b) $V_2 = \{f \in C^0 \text{ with } f : [0,1] \to \mathbb{R}\}$, i.e. the set of all continuous functions from $[0,1]$ to $\mathbb{R}$, with the standard addition and scalar multiplication.

   (c) $V_3 = \mathbb{R}^2$, with the standard addition and the following scalar multiplication:

$$\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda v_2 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

3. Let A be an $n \times n$ matrix and λ be an eigenvalue of A.

   (a) Show that λ^2 is an eigenvalue of A^2. More generally, are the eigenvalues of the product of two matrices the same as the product of the eigenvalues of the two matrices?

   (b) Show that $\lambda + \lambda$ is an eigenvalue of $A + A$. More generally, are the eigenvalues of the sum of two matrices the same as the sums of the eigenvalues of the two matrices?

Sub-section A.II: Answer at least one question from this sub-section

4. Answer both parts.

   (a) Let X denote a random variable with finite mean τ and variance σ^2. Show that for any $k > 0$

   $$P\{|X - \tau| \geq k\} \leq \frac{\sigma^2}{k^2}$$

   (b) A post office handles 10,000 letters per day with variance of 2,000 letters. What can be said about the probability that this post office handles between 8,000 and 12,000 letters tomorrow?
   What about the probability that more than 15,000 letters come in?

5. A research program studies the human health risk from recreational contact with water contaminated with pathogenic microbiological material. Water samples were taken from sites identified as having a heavy environmental impact from birds.

   (a) Let π be the true probability that a one-litre water sample from this type of site contains pathogenic material. What is the distribution of y, the number of samples containing the pathogenic material?

   (b) Out of $n = 116$ one litre water samples, $y = 17$ samples contained the pathogenic material. Using a *Beta* (1,2) prior for π, find $g(\pi|y)$, the posterior distribution of π given y.

   (c) Find the *normal* approximation to this posterior distribution.

   N.B. A random variable $X \sim Beta(\alpha, \beta)$ if its probability density function has the form

   $$f_X(x) \propto \left\{ \begin{array}{ll} x^{\alpha-1}(1-x)^{\beta-1} & 0 \leq x \leq 1 \\ 0 & elsewhere \end{array} \right\}$$

   Furthermore $E(X) = \frac{\alpha}{\alpha+\beta}$ and $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

6. Given a sample of size n from the Bernoulli (point binomial) distribution

   $$f(x; \theta) = \theta^x(1-\theta)^{1-x}, \qquad x = 0, 1,$$

   show that $\overline{X}$ is a sufficient statistic for θ. Is it efficient?

Sub-section A.III: Answer at least one question from this sub-section

7. Let $f : (1, \infty) \to \mathbb{R}$ be the function defined by $f(x) = (\ln x)^{\exp x}$. Let A and B be constants such that $y = Ax + B$ is the equation of the line tangent to the graph of this function at the point $(e, f(e))$, where $e = \exp(1)$ is the mathematical constant.
   (a) Evaluate A and B.
   (b) Draw this line on an xy-coordinate plane, and evaluate the area of the triangle formed by this line, the x-axis and the y-axis.

8. Suppose that temperature varies continuously on a circle. Show that there must be two antipodal points on this circle at which the temperature is identical.

9. There are two firms A and B who engage in quantity competition, with firm A choosing production quantity first.
   (a) Firm B chooses a production quantity r (having observed firm A's production quantity q) to maximize profits $\Pi^B = (10 - q - r)\, r - \delta r$. Assume that there is a strictly positive solution $r^*(q, \delta)$. Show that $\Pi^B_{rr} < 0$ and find $\frac{\partial r^*}{\partial q}$ and $\frac{\partial r^*}{\partial \delta}$.
   (b) Knowing that firm B will respond as above, firm A chooses q to maximize $\Pi^A = (10 - q - r^*(q, \delta))\, q - c(q, \gamma)$, where $c_q > 0$, $c_{qq} > 0$, and $c_{q\gamma} < 0$. What might γ represent? Assume that there is a strictly positive solution $q^*(\gamma, \delta)$. Show that $\Pi^A_{qq} < 0$ and find the signs of $\frac{\partial q^*}{\partial \gamma}$ and $\frac{\partial q^*}{\partial \delta}$.

SECTION B

Sub-section B.I: Answer only one question from this sub-section

10. Answer all parts.

(a) Let V be a vector subspace of $\mathbb{R}^n$ with an orthogonal basis $\{v_1, ..., v_p\}$, and let $\{w_1, ..., w_q\}$ be an orthogonal basis of $V^{\perp}$. Show that the set $\{v_1, ..., v_p, w_1, ..., w_q\}$ is an orthogonal set.

(b) Let

$$\tilde{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \tilde{v}_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

and let $\tilde{V} = Span\,\{\tilde{v}_1, \tilde{v}_2\}$. Find a set of vector(s) $\{\tilde{w}_1, ..., \tilde{w}_q\}$ that span $\tilde{V}^{\perp}$.

(c) Is the set $\{\tilde{v}_1, \tilde{v}_2, \tilde{w}_1, ..., \tilde{w}_q\}$ an orthogonal basis for $\mathbb{R}^3$? Explain your reasoning carefully. Comment on the relationship of the dimensions of $\tilde{V}$, $\tilde{V}^{\perp}$ and $\mathbb{R}^3$.

11. Imagine an online retailer with two types of customers: *one-time* customers and *repeat* customers. Let the number of one-time and repeat customers in month t be o_t and r_t, respectively. Any given customer is either a one-time or a repeat, but no customer is both in a given month. A one-time customer can become a repeat customer. Data from the retailer shows that each month 40% of one-time customers remain one-time customers (i.e. they have not placed any additional orders since the first time they have used the retailer) and that there is a 90% chance that repeat customers continue to buy the retailer's goods.

(a) Write the evolution of the retailer's customers over time as a system of difference equations.

(b) Show that in the long run this retailer will not have enough business to survive.

(c) The retailer is now considering introducing a referal incentive scheme that will encourage its repeat customers to get a friend of theirs to try the retailer. What is the minimum per-month referal success rate ρ from the repeat customers that will keep the retailer afloat in the long run? What proportion of one time to repeat customers will the retailer have in the long run for such a referal rate ρ?

Sub-section B.II: Answer only one question from this sub-section

12. Let E, C, and M, represent three binary random variables denoting, respectively, recovery from an illness, whether a drug was taken, and gender. Eighty individuals suffering from a given disease were randomly sampled from a population of sufferers. Table 1 (on the next page) shows some summary statistics on the joint distribution, $f(E, C, M)$, by treatment, recovery and gender.

   (a) We observe that

   $$\begin{aligned} \widehat{\mu}_{E|C} &> \widehat{\mu}_{E|\bar{C}} \\ \widehat{\mu}_{E|C,M} &< \widehat{\mu}_{E|\bar{C},M} \\ \widehat{\mu}_{E|C,\bar{M}} &< \widehat{\mu}_{E|\bar{C},\bar{M}} \end{aligned}$$

   where $\widehat{\mu}_{E|C} > \widehat{\mu}_{E|\bar{C}}$ indicates that recovery rate for patients taking the drugs exceeds that of the control. Explain these paradoxical results.

   (b) The joint distribution of the two random variables E and C is:

   $$f_{E,C}(e, c) = f_{E|C}(e|c)f_C(c)$$

   By considering this joint distribution in conjunction with the full distribution $f(E, C, M)$, demonstrate that the paradox arises from a problem of endogeneity.

   (c) An analyst has access to the data on the eighty individuals that were used in the study. In examining the relationship between recovery and treatment he specifies the following regression equation:

   $$E_i = \alpha + \beta C_i + \varepsilon_i,$$

   α and β are unknown parameters and $\varepsilon_i \sim i.i.d(0, \sigma_2)$. How might this equation be modified to control for endogeneity?

Table 1: Combined (Males and Females)

	E	$\bar{E}$	Total
Drug (C)	20	20	40
$\bar{C}$	16	24	40

Males

	E	$\bar{E}$	Total
Drug (C)	18	12	30
$\bar{C}$	7	3	10

Females

	E	$\bar{E}$	Total
Drug (C)	2	8	10
$\bar{C}$	9	21	30

13. The pdf for a Pareto distribution is

$$f(y) = \alpha y^{-(\alpha+1)}, \quad 0 < \alpha < \infty,\ y \geq 1$$

(a) Obtain the quantile function. What is the value of the median when $\alpha = 1$?

(b) Show that the mean only exists for $\alpha > 1$. Given a random sample of size n, find a simple method of moments estimator for α.

(c) Derive the ML estimator for α and find its asymptotic variance. What advantages might the ML estimator have over the simple MM estimator?

(d) Suppose we reparameterize by setting $\alpha = \exp(\theta)$ and maximize the likelihood function with respect to θ, where $-\infty < \theta < \infty$. What is the 95% confidence interval for θ?

(e) Construct a (large sample) likelihood ratio (LR) test of the null hypothesis that $\alpha = 1$ against the alternative $\alpha \neq 1$. Would the LR test of $\theta = 0$ give the same result?

(TURN OVER)

Sub-section B.III: Answer only one question from this sub-section

14. Answer all parts.
   (a) What does it mean for a production function $f : \mathbb{R}^3 \to \mathbb{R}$ with three inputs to have *constant returns to scale*? Define carefully.
   (b) Suppose that $g : \mathbb{R}^3 \to \mathbb{R}$ has continuous partial derivatives with respect to all three of its variables. If x, y, and z are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, then express $\frac{d}{dt} g(x(t), y(t), z(t))$ using the chain rule.
   (c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a *constant returns to scale* production function with three inputs: *capital*, *labour*, and *technology.* Prove that

$$f_{21}(K, L, T)K + f_{22}(K, L, T)L + f_{23}(K, L, T)T = 0.$$

   Recall that f_i denotes the partial derivative of f with respect to its ith variable.

15. Answer all parts.
   (a) Find the solution to the differential equation $\frac{d^2y}{dx^2} = y$ which satisfies the initial conditions $y(0) = 3$ and $y'(0) = 1$.
   (b) This differential equation can be written in the form $z' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z$ where $z = \binom{y}{y'}$. The Euler approximation with step size δ to $z(\delta n)$ is $z_n = \binom{y_n}{y'_n}$, where $z_0 = \binom{3}{1}$ and $z_{n+1} = z_n + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z_n \delta$. Here y_n is the approximation to $y(\delta n)$ and y'_n is the approximation to $y'(\delta n)$. Find z_n and hence the approximation y_n to $y(\delta n)$.
   (c) Given that $\lim_{x \to 0} \frac{\log(1+\alpha x)}{x} = \alpha$, find $\lim_{n \to \infty} n \log\left(1 + \frac{\alpha}{n}\right)$ and hence show that $\left(1 + \frac{\alpha}{n}\right)^n \to e^{\alpha}$, as $n \to \infty$.
   (d) Given a particular value of $x > 0$, give the Euler approximation to $y(x)$ using n steps of size $\delta = \frac{x}{n}$. Show that this converges to $y(x)$ as $n \to \infty$.

END OF PAPER