PRELIMINARY EXAMINATION FOR PART II OF THE ECONOMICS TRIPOS

Tuesday 8 June 1999 9-12

Paper 6

MATHEMATICS

The paper consist of two Sections; *A and B.*

Candidates may attempt ***SIX*** *questions from Section A, and* ***THREE*** *questions from Section B.*

Credit will be given for complete answers; *answers to individual parts of questions will gain less than pro rata credit.*

Write on **one** *side of the paper only*.

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Section A

A1 Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A2 Consider the following simple financial model. There are three states and three assets. Asset 1 pays 3 in state 1, nothing in state 2 and 1 in state 3. Asset 2 pays nothing in state 1, 5 in state 2 and 2 in state 3. Asset 3 pays 6 in state 1, 5 in state 2 and 4 in state 3.

1. Construct the return matrix and find a basis for, and the dimension of, the subspace spanned by the assets.
2. Given an arbitrary payoff vector $\mathbf{b} \in \Re^3$, does there necessarily exist a portfolio which will achieve $\mathbf{b}$? Explain your answer.
3. Find a basis for the subspace consisting of all those portfolios which deliver a zero payoff in each state.

A3 Suppose that a population is divided into three classes, "poor", "middle class" and "rich". Of the poor, one quarter become middle class and one quarter become rich in each period. Of the middle class, one third become rich in each period while none become poor. The rich always stay rich. In the initial period everyone is poor. Find an expression for the population distribution after t periods. What is the distribution in the limit as $t \to \infty$?

A4 The joint probability density function of the random variables X and Y is given by

$$f_{X,Y}(x,y) = \left\{ \begin{array}{cc} c(x-y) & x = 1,2 \quad y = 0,1 \\ 0 & \text{otherwise} \end{array} \right\}.$$

1. What is the value of c?
2. Calculate the marginal density functions $f_X(x)$ and $f_Y(y)$.
3. Calculate the conditional density function $f_{X|Y}(x|y = 0)$.
4. Are X and Y independent?

A5 Let $Y = (Y_1, Y_2, \ldots, Y_N)$ be a random sample from the uniform probability density function (p.d.f.)

$$f(y;\theta) = (1-\theta)^{-1}, \quad \theta \le y \le 1$$

1. Show that if Y has p.d.f. $f(y;\theta)$ then $E(Y) = (1+\theta)/2$.
2. Prove that the estimator $\widehat{\theta} = 2(\sum_i y_i/N) - 1$ is an unbiased estimator of θ.
3. Prove that $Var(\widehat{\theta}) = (1-\theta)^2/3N$.

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A6 The entire output of a factory is produced on three machines which account for 20%, 30% and 50% of the output, respectively. The fraction of defective items produced is 5% for the first machine, 3% for the second, and 1% for the third.

1. What fraction of the total output is defective?

2. If an item is chosen at random from the total output and is found to be defective, what is the probability that it was made by the third machine?

A7 Verify that $\left(-\frac{2}{3}, \frac{1}{3}, -\frac{4}{3}\right)$ is a local minimum of the function

$$5x^2 + 2y^2 + 2z^2 - 2xy + 2yz - 4xz + 2x + 2z.$$

A8

1. Find and classify the equilibrium point of the differential equation

$$\frac{dx}{dt} = 1 - x^3.$$

2. Find and classify the equilibrium point of the difference equation

$$x_{n+1} = 2 - x_n^3.$$

A9 The expenditure function $e(\mathbf{p}, u)$ is given by

$$e(\mathbf{p}, u) = 2u(p_1 p_2)^{\frac{1}{2}}.$$

Use Taylor series to approximate, to quadratic order in the small price change (h, k), the change in the expenditure function when prices change from $\mathbf{p} = (4, 9)$ to $\mathbf{p} = (4 + h, 9 + k)$.

Section B

B1 A firm has a production process which produces two (joint) outputs from three inputs. If the input quantities are x_1, x_2 and x_3, the resulting output is $3x_1 + x_2 + 3x_3$ units of output good 1 and $4x_1 + x_2 + x_3$ units of output good 2. The firm has to produce, in the least costly fashion, at least 2 units of output good 1 and at least 1 unit of output good 2. The prices of the inputs are respectively 12, 2 and 3.

1. Set up the firm's problem as a linear programming problem.

2. Write down the dual problem and solve it.

3. Using complementary slackness principles, solve the original problem. Verify that the duality theorem holds.

4. How much would the minimum cost fall if the required amount of output good 1 were reduced by a small amount?

B2

1. Show that, if the rank of an $m \times n$ matrix A is r, then the dimension of its nullspace is $n - r$.

2. Given a subspace $V \subseteq \Re^n$ of dimension k, define the orthogonal complement of V and show that it has dimension $n - k$.

3. Given matrices A and B such that BA exists, show that $Null(A) \subseteq Null(BA)$. Hence show that, if AA has an inverse, then so does A.

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B3 Let $X_1, ..., X_n$ denote a random sample from the Poisson distribution with probability density function

$$f(x) = \frac{\lambda^x \exp(-\lambda)}{x!} \quad \lambda > 0 \quad x = 0, 1, 2, 3, ...$$

1. For what type of phenomenon would a Poisson process constitute a valid probability model?
2. Find the maximum likelihood estimator (MLE) of λ.
3. Show that $\hat{\lambda}$ is unbiased for λ.
4. Using the invariance property of MLE, find the MLE for $\beta = e^{-\lambda}$ and show that $\hat{\beta}$ is biased for β. Can you say anything about the consistency properties of $\hat{\beta}$?

B4

1. A random variable X has density function $f(x)$. Define the moment generating function $M_X(t)$.
2. If X and Y are independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$ derive the moment generating function of $\alpha X + \beta Y$ where α and β are constants.
3. A random variable X takes the values 1 with probability p and 0 with probability $1 - p$. What is the moment generating function of X?
4. $\{X_i \mid i = 1, ..., n\}$ are independent random variables each with the same distribution as X. Let $S_n = \frac{1}{n}(X_1 + X_2 + ...X_n)$. Calculate the moment generating function of S_n and hence find its mean and variance.

B5 [***In this question, you may quote without proof any results from consumer theory you require.***]

1. Explain briefly the origin of the equation

$$\mathrm{V\ big(}\ \mathbf{p}, \mathrm{e}(\mathbf{p}, u)\ \mathrm{big)} \ = u,$$

where $\mathrm{V}(\mathbf{p}, \mathrm{m})$ is the indirect utility function and $\mathrm{e}(\mathbf{p}, u)$ is the expenditure function derived from a (direct) utility function $\mathrm{U}(\mathbf{x})$.

2. Obtain Roy's identity for the Marshallian demand function $\mathbf{D}(\mathbf{p}, m)$:

$$\mathrm{D}_i(\mathbf{p}, m) = -\frac{\partial \mathrm{V\ big/}\ \partial p_i}{\partial \mathrm{V\ big/}\ \partial m}.$$

Hence prove that $\mathrm{V}(t\mathbf{p}, tm) = \mathrm{V}(\mathbf{p}, m)$, for any scaling factor t, and explain this result by means of a diagram.

3. An indirect utility function is given by

$$\mathrm{V\ big(}\ \mathbf{p}, m\ \mathrm{big)} \ = \frac{m^4}{p_1^3 p_2}.$$

Find the expenditure function, the compensated demand functions and the Marshallian demand functions. Verify that your demand functions have the correct scaling behaviour.

4. Find also the direct utility function.

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B6

1. Write down the definition of the term ***concave*** as applied to the function of n variables $\mathrm{f}(\mathbf{p})$. The expenditure function $\mathrm{e}(\mathbf{p}, u)$ is defined by

$$\mathrm{e}(\mathbf{p}, u) = \mathbf{p} \bullet \mathbf{x}^*, \ \mathrm{mbox}\ \mathbf{p} \bullet \mathbf{x}^* \leq \mathbf{p} \bullet \mathbf{x} \quad \forall \mathbf{x}.$$

By considering price vectors $\mathbf{p_1}$, $\mathbf{p_2}$ and an intermediate price vector $\mathbf{p}$, with corresponding optimal demand bundles $\mathbf{x}_1^*$, $\mathbf{x}_2^*$ and $\mathbf{x}^*$, show that $\mathrm{e}(\mathbf{p}, u)$ is concave in prices.

2. Use matrix methods to solve the difference equations

$$x_{n+1} = x_n + hy_n$$

$$y_{n+1} = hx_n + y_n$$

subject to the conditions $x_0 = 0$ and $y_0 = 2$.

3. Find the limit of your solution as $h \to 0$ and $n \to \infty$, with $nh = t$ for fixed t. To what system of differential equations do these equations correspond in the limit $h \to 0$?