PRELIMINARY EXAMINATION FOR PART II OF THE ECONOMICS TRIPOS

Tuesday 13 June 2000 9-12

Paper 6

MATHEMATICS

The paper consist of two Sections, A and B.

Candidates may attempt **SIX** *questions from Section A, and* **THREE** *questions from Section B.*

Credit will be given for complete answers; answers to individual parts of questions will gain less than pro rata credit.

Write on **one** *side of the paper only.*

(TURN OVER

Section A

A1 For what values of a and b, if any, does the system of equations

$$\left\{ \begin{array}{c} x + 3y - 2z = b \\ 3x + ay - 6z = 3 \end{array} \right\}$$

have

(a) no solution,

(b) a unique solution and

(c) many solutions?

If there are many solutions, what is the dimension of the set of solutions?

A2 Find a basis for, and the dimension of, the set of all vectors in four-dimensional space such that the first element is equal to the second element and three times the third element. Find a basis for, and the dimension of, the orthogonal complement of this set.

A3 Three individuals have one unit of a good to share between them. Each has a linear utility function:

$$u_1(x_1) = 3x_1, \quad u_2(x_2) = 2x_2, \quad u_3(x_3) = x_3,$$

where x_i is the quantity of the good given to individual i and $u_i(\cdot)$ is i's utility function. They decide to share the resource in such a way as to maximise the sum of their utilities. Formulate this as a linear programming problem and solve it via its dual. Verify that the duality theorem holds. What is the interpretation of the dual variable?

A4 Consider the following distribution for a random variable x.

x	$\sqrt{3}$	$-\sqrt{3}$	0
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{4}{6}$

Show that x has moments which include the first five moments of $Z \sim N(0,1)$.

A5 Consider the simple one-parameter Normal Model

$$X_i = \mu + \varepsilon_i,$$

where the probability density function is given by

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x-\mu)^2\}, \theta = \mu \in \Re, x \in \Re.$$

(a) Based upon a random sample of size n, find the expectation and variance of the following estimators of μ:

i. $\widehat{\mu}_1 = X_1$,

ii. $\widehat{\mu}_2 = \frac{1}{2}(X_1 + X_2)$,

iii. $\widehat{\mu}_3 = (X_1 - X_n)$ and

iv. $\widehat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(b) Derive the sampling distribution, and in doing so write down the finite sample properties of each estimate.

(c) Which estimators are consistent?

A6 Consider the simple Poisson model

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}, \theta \in (0, \infty),\ \ x = 0, 1, 2, 3, ...$$

(a) Find the likelihood, log-likelihood and the maximum likelihood estimate $\widehat{\theta}_n$ of θ.

(b) Based upon the hypothesis test

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned}$$

and significant level α, compute the likelihood ratio and the asymptotic likelihood ratio test statistic.

(c) Consider the probability models

$$f(x;\boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta$$

and

$$g(x;\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \daleth$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$**,** respectively, denote vectors of parameters with parameter space Θ and $\daleth$. If $f(x;\boldsymbol{\theta})$ and $g(x;\boldsymbol{\lambda})$ are non-nested, in what sense is the likelihood ratio test an inappropriate test statistic?

A7 Consider the differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = -4$$

with initial conditions $y(0) = 0$, $\frac{dy}{dt}(0) = k$. For what values of k does y converge to a steady state?

A8 Consider a production function of the form

$$F(K, L) = \left(K^{-a} + L^{-a}\right)^{-\frac{1}{a}}.$$

(a) Is this function homogeneous?

(b) Does it display increasing, constant or decreasing returns to scale?

(c) Let G be some function. Find an approximation to $G(K + g, L + h)$ by taking a first-order Taylor expansion of G about (K, L).

(d) Show using Euler's theorem for homogeneous functions that this approximation is exact when G is the function F given above, $g = K$ and $h = L$.

A9 Consider an agent with utility function $U(x, y) = Ax^a y^{1-a}$ and wealth m.

(a) Write the Lagrangean of his expenditure minimisation problem.

(b) Find his compensated demand function $\mathbf{C}(\mathbf{p}, u)$ and his expenditure function $e(\mathbf{p}, u)$.

Section B

B1 Let A be a symmetric $n \times n$ matrix.

(a) Define

i. positive definiteness and

ii. positive semi-definiteness.

(b) Using the diagonalisation formula for A, show that A is positive definite if all its eigenvalues are strictly positive.

(c) For what range of values of a is the matrix

$$\begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}$$

positive definite?

(d) If, in the matrix

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix},$$

b is strictly positive and $ab > 1$, is the matrix positive definite? What if $ab = 1$?

B2 Consider a Leontief economy with n industries and no non-produced factors.

(a) Explain the meaning of the element a_{ij} of the input-output matrix A and write down the relation between final demand d and total output x.

(b) Show that, if v is an eigenvector of A with eigenvalue λ, then v is also an eigenvector of $(I - A)^{-1}$ if the inverse exists. What is the associated eigenvalue?

(c) Assume that every element of A is strictly positive. This implies that the largest eigenvalue of A, λ_1, is strictly positive and the associated eigenvector, v_1, is strictly positive.

i. Show that if $\lambda_1 = 1$ then $(I - A)$ is singular.

ii. Show that if $\lambda_1 > 1$ then $(I - A)^{-1}$, if it exists, must have some negative elements. Is it possible in this case to produce v_1 as a final demand vector?

iii. Show that if $\lambda_1 < 1$ then $A^n v_1$ converges to zero as $n \to \infty$. Why does this imply that A^n converges to the zero matrix? Show that in this case $(I - A)^{-1}$ exists and is the sum of non-negative matrices.

B3 Based on the statistical model

$$X_i = \mu + \varepsilon_i, \varepsilon_i \sim N(0, \sigma^2)$$

with likelihood

$$L(\boldsymbol{\theta}_i; \mathbf{x}) \propto \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}:$$

(a) Derive maximum likelihood estimators of μ and σ^2.

(b) Determine the Fisher information matrix

$$I_n(\boldsymbol{\theta}) = E\left[-\frac{d^2}{d\theta^2}\ln L(\boldsymbol{\theta}; x)\right]$$

where $\boldsymbol{\theta} = \{\mu, \sigma^2\}$.

(c) Using results from part (b), find the Cramer-Rao lower bound for μ and σ^2, and thereby demonstrate that

$$Var[\overline{x}, s^2] - [I_n(\boldsymbol{\theta})]^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 2\sigma^4/(n(n-1)) \end{bmatrix}$$

B4 (a) Compare and contrast the Fisher and Neyman-Pearson approaches to hypothesis testing.

(b) What do you understand by a uniformly most powerful test.

(c) Explain the notion of

i. an unbiased test and

ii. a consistent test.

(d) Describe the Fisher procedure for testing a hypothesis of the form

$$H_0 : \theta \in \Theta_0,$$

where Θ_0 denotes the parameter space under the null.

B5 Consider the following non-linear system of differential equations:

$$\begin{aligned} \dot{x} &= x^2 + (2+\alpha)x + y \\ \dot{y} &= \alpha x + y + 3 \end{aligned}$$

(a) Find the fixed points of the system.

(b) Find a linear approximation to the system about the fixed point at which $x = 1$.

(c) Find the values of α for which this approximation converges to a steady state and has solutions of the form $Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ where λ_1 and λ_2 are real numbers.

(d) Is this fixed point stable for these values of α?

B6 A consumer has demand given by

$$D(\mathbf{p}, m) = \left(\frac{p_1 - p_2 + m}{2p_1}, \frac{p_2 - p_1 + m}{2p_2} \right)$$

and indirect utility given by

$$V(\mathbf{p}, m) = \frac{m - p_1 - p_2}{2(p_1 p_2)^{\frac{1}{2}}}.$$

(a) Express the Lagrange multiplier of the consumer's utility maximisation problem as a function of (p_1, p_2), and explain its economic significance.

(b) Find an expression for the compensated demand function $\mathbf{C}(\mathbf{p}, u)$.

(c) Find the expenditure function $e(\mathbf{p}, u)$ and verify that:

$$\mathbf{D}(\mathbf{p}, e(\mathbf{p}, u)) = \mathbf{C}(\mathbf{p}, u).$$

(d) For what value of c is the function

$$\begin{aligned} U(x, y) &= (x - c)^{\frac{c}{2}} (y - c)^{\frac{c}{2}} \qquad (*) \\ x, y &\geq c \end{aligned}$$

the utility function of this consumer?

(e) Define concavity of $U(x, y)$ and check whether (*) is concave for all c. (You may use the result that a function f has a negative semi-definite Hessian if and only if it is concave.)