PRELIMINARY EXAMINATION FOR PART II OF THE ECONOMICS TRIPOS

Tuesday 12 June 2001 9-12

Paper 6

MATHEMATICS

The paper consists of two Sections, A and B

Candidates may attempt **six** questions from Section A and **three** questions from Section B. Each section carries 50% of the total marks of this paper.

Credit will be given for complete answers; answers to individual parts of questions will gain less than pro rata credit.

Write on **one** side of the paper only.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$.

a. Define and give a geometric description of $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

b. Determine whether $\mathbf{w}$ belongs to $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

2. **a.** Let r be an eigenvalue of an invertible matrix $\mathbf{A}$. Show that r^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.

b. Let $\mathbf{A}$ be an invertible matrix such that $\mathbf{A}^{-1} = \mathbf{A}^{\top}$. Show that every eigenvalue of $\mathbf{A}$ must be either 1 or -1.

c. Provide an example of an invertible 2×2 matrix $\mathbf{A}$ such that every entry of $\mathbf{A}$ is a real number and every eigenvalue of $\mathbf{A}^2$ is negative.

3. Consider the following quadratic form for $\mathbf{x} = (x_1, x_2, x_3)^{\top}$:

$$Q(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3.$$

a. Find the 3×3 symmetric matrix $\mathbf{A}$ such that $\mathbf{x}^{\top}\mathbf{A}\mathbf{x} = Q(\mathbf{x})$.

b. Determine whether Q is positive definite. (*Hint*: One of the eigenvalues of $\mathbf{A}$ is equal to 2.)

4. Find and classify the stationary points of the function

$$f(x,y) = x^3 + xy^2 - 12x - y^2$$

5. **a**. Find the solution of

$$\ddot{y} - \dot{y} - 2y = 5e^{-3t}$$

with initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$.

b. Find the solution of

$$x_{n+2} - 9x_n = 1$$

with initial conditions $x_0 = 0$ and $x_1 = 2$.

6. Consider the following non-linear differential equations:

$$\dot{x} = (4 - x - y)x$$

$$\dot{y} = (6 - 3x - y)y$$

a. Determine the equilibrium points.

b. Using a linear approximation, determine the stability of these equilibrium points.

7. Define

$$f(x) = \begin{cases} 1 - |x| & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

a. Sketch this function and show that it is a valid probability density function.

b. Derive its moment generating function.

c. Find its mean and variance.

8. For the following one parameter exponential model

$$f(x;\mu) = \frac{1}{\mu}\exp\left(-\frac{x}{\mu}\right),\ \mu > 0,\ x > 0,$$

determine the likelihood function and the log-likelihood function, and show that the maximum likelihood estimator for μ is unbiased.

9. Consider the simple Bernoulli Model

$$f(x;\theta) = \theta^x(1-\theta)^{1-x},\ 0 \leq \theta \leq 1, x = 0, 1.$$

Based upon a random sample $\mathbf{X} = (X_1, \cdots, X_n)$, where $n > 3$, consider the following estimators of θ:

$$\begin{aligned}
\widehat{\theta}_1 &= X_1, \\
\widehat{\theta}_2 &= \frac{1}{2}(X_1 + X_2), \\
\widehat{\theta}_3 &= \frac{1}{3}(X_1 + X_2 + X_3), \\
\widehat{\theta}_n &= \frac{1}{n}\sum_{i=1}^{n} X_i, \\
\widetilde{\theta} &= \frac{1}{n+1}\sum_{i=1}^{n} X_i
\end{aligned}$$

a. Write down the sampling distribution for each of these estimators.

b. Determine the bias of each estimator and indicate which ones are consistent.

SECTION B

1. Four goods, G, H, I, J, are produced under constant returns to scale. The production of one unit of good G requires 0.9 units of good G, 0.2 units of good H, and 0.1 units of good J. A unit production of good H requires 0.1 units of good I, and so does a unit production of good I. A unit production of good J requires 0.2 units of good G, 0.1 units of good H, 0.3 units of good I, and 0.8 units of good J. It is impossible to substitute one input for another in production.

- **a.** Write down the input matrix $\mathbf{A}$.
- **b.** Denote the 4×4 identity matrix by $\mathbf{I}$. Find a basis for the column space of $\mathbf{I} - \mathbf{A}$. What is the dimension of the column space?
- **c.** Find a basis for the row space of $\mathbf{I} - \mathbf{A}$.
- **d.** Write down the definition for $\mathbf{A}$ to be productive and then determine whether $\mathbf{A}$ is productive.

2. The numbers of wolves and rabbits in each year $t = 0, 1, 2, \cdots$ is denoted by the vector $\mathbf{x}_t = (w_t, r_t)^\top$. Suppose that

$$\begin{aligned} w_{t+1} &= 0.6w_t + 0.8r_t, \\ r_{t+1} &= -0.15w_t + 1.4r_t, \end{aligned}$$

for every $t = 0, 1, 2, \cdots$. Assume that the initial number of wolves is 20 and the initial number of rabbits is 11, so that $\mathbf{x}_0 = (20, 11)^\top$.

- **a.** Write down the numbers of wolves and rabbits in year t in terms of t.
- **b.** Estimate the long-term rate of growth or decay of the populations and the eventual ratio of wolves to rabbits.

3. A consumer has a utility function $U(\mathbf{x})$ which is concave and homogeneous of degree n in $\mathbf{x}$.

a. Prove that each partial derivative $\partial U(\mathbf{x})/\partial x_i$ is homogeneous of degree $n - 1$.

b. Denote by $\lambda(\mathbf{p}, u)$ the Lagrange multiplier for the expenditure minimisation problem under a given price vector $\mathbf{p}$ and a given utility level u. Show that $\lambda(\mathbf{p}, u)$ is homogenous in u and identify the degree of homogeneity.

4. **a**. For positive constants $A,\ B,\ \alpha,$ and $\beta,$ the production function $F(K, L)$ is defined by

$$F(K, L) = (AK^{\alpha} + BL^{\alpha})^{\beta}.$$

i. Show that it has a constant elasticity of substitution.

ii. Show that it is homogeneous and find the degree of homogeneity.

iii. Is it concave if $\alpha < 1/\beta$?

b. The Iceland Sheep Company is a monopolist in the production of both wool and mutton. The company buys sheep from the farmers at a fixed price of 10 krones per head. Each sheep is then shaved to produce 1 kilo of wool and butchered to produce 25 kilos of mutton. Neither shaving nor butchering requires any additional cost. When q_w kilos of wool are sold, the company's average revenue (price) per kilo is equal to

$$p_w = 100 - 6\sqrt{q_w}\,.$$

When q_m kilos of mutton are sold, the average revenue (price) per kilo is equal to

$$p_m = 200 - \frac{q_m}{25}\,.$$

Assume that the sales of wool does not affect the average revenue of mutton and that the sales of mutton does not affect the average revenue of wool. Assume also that the company must sell all the wool and mutton it can produce out of the sheep it has bought from the farmers.

(Turn over for continuation of question 4

   i. What is the company's profit if it buys x sheep?
   ii. Show that the company's profit is maximised when it buys 100 sheep.
   iii. What would happen to the company's profit if it could sell less than the wool and/or mutton it can produce?

5. Let X be a random variable with a probability density function $f(x)$.
   a. Define its central and non-central moments.
   b. Show that if μ_r is the r-th central moment and μ'_r the r-th non-central moment, then
   $$\mu_2 = \mu'_2 - \mu^2,$$
   $$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3,$$
   where $\mu = E(X)$.
   c. Hence show that if $f(x)$ is symmetric (i.e., $f(\mu - z) = f(\mu + z)$ for all z) and $\mathrm{Var}(X) = \sigma^2$, then
   $$E(X^3) = 3\mu\sigma^2 + \mu^3.$$

6. Answer two out of the following three questions
   a. Distinguish the notion of *p-values* in the context of a Fisher test and a significance level in the context of the Neyman-Pearson approach to hypothesis testing.
   b. With reference to an appropriate example, explain what you understand by a pivotal quantity.
   c. Show that the Mean-Square Error of an estimator is equal to the sum of its variance and squared bias. What are the advantages and disadvantages of the Mean-Square Error criterion in evaluating an estimator?

END OF PAPER