PRELIMINARY EXAMINATION FOR PART II OF THE ECONOMICS TRIPOS

WEDNESDAY 12 JUNE 2002 9 to 12

Paper 6

MATHEMATICS

The paper consist of two Sections; A and B.

Each Section carries 50% of the total marks

*Candidates may attempt **SIX** questions from Section A,*
*and **THREE** questions from Section B.*

Credit will be given for complete answers; answers to individual parts of questions will gain less than pro rata credit.

*Write on **one** side of the paper only.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Section A

A1 Let t be a real number. Consider the symmetric matrix

$$\begin{pmatrix} 1+t & 1 & 1 \\ 1 & 1+t & 1 \\ 1 & 1 & 1+t \end{pmatrix}$$

Find all values of t for which it is positive definite and all values of t for which it is positive semi-definite. Justify your answers BOTH in terms of principal minors AND in terms of eigenvalues.

A2 Let inverse demand be given by $P = a - bQ$ and let supply be given by $P = c + dQ$ where a, b, c, d are strictly positive parameters. Use Cramer's rule to solve for an equilibrium. What restrictions on the parameters are necessary so that an equilibrium exists? What restrictions on the parameters are necessary so that the equilibrium is unique? Carefully explain your answer.

A3 Let $\mathbf{A} \neq \mathbf{0}$ be a (2×2) matrix. Is it possible that $\mathbf{A}^k = \mathbf{0}$ for some k? Give an example or prove that it is impossible.

A4 The probabilities assigned to events on a sample space Ω satisfy the following properties:

i. $P(E) \geq 0$ for every $E \subset \Omega$

ii. $P(\Omega) = 1$

iii. if $E \subset F \subset \Omega$, then $P(E) \leq P(F)$

iv. if A and B are disjoint subsets of Ω, then $P(A \cup B) = P(A) + P(B)$

Provide a proof of the above.

A5 (a) Let S_n be the number of successes in n Bernoulli trials with probability p for success on each trial. Show using Chebyshev's inequality that for any $\epsilon > 0$

$$P(|\frac{S_n}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}$$

(b) Let X be any random variable which takes on values $0, 1, 2, ..., n$ and has $E(X) = Var(X) = 1$. Show that for any integer k

$$P(X \geq k+1) \leq \frac{1}{k^2}$$

A6 (a) Prove that for any three events A, B, C, each having positive probability

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

(b) Prove that if A and B are independent so are

i. A and $\tilde{B}$

ii. $\tilde{A}$ and $\tilde{B}$

where $\tilde{A}$ denotes the complement of A.

(c) Prove that if $P(A|C) \geq P(B|C)$ and $P(A|\tilde{C}) \geq P(B|\tilde{C})$ then $P(A) \geq P(B)$.

A7 (a) Find the solution of

$$\ddot{y} + 5\dot{y} - 6y = 24e^{-2t}$$

with initial condition $y(0) = 0$ and $\dot{y}(0) = -15$

(b) Find the solution of

$$x_{n+2} - 25x_n = -1$$

with initial condition $x_0 = 0$ and $x_1 = 11/24$

A8 Find and classify the stationary points of the function

$$f(x, y) = xy(3x + 6y - 2)$$

A9 Given the following non-linear differential equations

$$\begin{aligned} \dot{x} &= 2xy - x^2 \\ \dot{y} &= 2 - xy \end{aligned}$$

(a) Determine the equilibrium points.

(b) Determine the stability of each of the equilibrium points.

Section B

B1 An economy has two sectors, A and B. Both sectors use a non-produced factor, labour. The input-output matrix for this economy is

$$\begin{pmatrix} 0.6 & 0.2 \\ 0.1 & 0.7 \end{pmatrix}$$

where the first row refers to sector A and the second to B. The vector of labour requirements is $\mathbf{l}^{\top} = \begin{pmatrix} 0.1 & 0.2 \end{pmatrix}$.

(a) If the final demand vector is $(1, 1)$ for goods A and B, find the gross outputs of the two sectors and the demand for labour.

(b) If the nominal wage rate is 1 and there are zero profits, what are the prices of the two goods?

(c) When is an input-output matrix called productive? Provide an example of a (2×2) input-output matrix $\mathbf{A}$ which is such that $(\mathbf{I} - \mathbf{A})$ is invertible, but $\mathbf{A}$ is not productive.

B2 Consider two companies, each selling one rival branded product. Each year, 70% of customers who bought product 1 the previous year buy product 1 this year, 30% switch to product 2. In the case of product 2, 80% of customers who bought it last year continue to buy it this year, while 20% switch to product 1. This information can be summarised in the following matrix:

$$\mathbf{A} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$$

where the entries in the ith column show the fraction of purchasers of product i this year who buy each product next year. In each year the customers held by each firm can be expressed by a column vector $\mathbf{x}_t$. Assume that $\mathbf{x}_0 = \begin{pmatrix} 10 & 20 \end{pmatrix}^{\top}$.

(a) Carefully explain why

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t.$$

(b) Find a formula for $\mathbf{x}_t$ as a function of t only.

(c) Using your answer to (b), examine the allocation of the number of customers to each firm in the long run.

B3 (a) Explain what you understand by the following statements:

(i) Y is a statistic; the statistic Y is an unbiased estimator of θ.

(ii) the statistic Y is a fully efficient estimator.

(b) Let the statistic Y be an unbiased estimator for θ, with variance $k\theta^2$. Define the mean square error $MSE(X)$ of any estimator X of θ as

$$MSE(X) = E[(X - \theta)^2].$$

Find $MSE(cY)$, where c is some constant. For what value of c is $MSE(cY)$ a minimum?

B4 Let $X_1, X_2, ..., X_n$ denote a random sample of size n where each X_i has probability density function given by

$$f(x;\theta) = \frac{1}{\theta}\exp\{-\frac{x}{\theta}\}, \quad \theta > 0, \ x > 0$$

(a) Find the distribution of the sample.

(b) Find the log-likelihood function and show that the maximum likelihood estimator $\hat{\theta}$ is unbiased for θ.

(c) Can we say that $\hat{\theta}$ is a fully efficient estimator?

B5 The demand function for a certain good is $D(p, m)$ and the supply function is $S(p)$. For a given m, the equilibrium price p^* is given by $p^* = f(m)$.

(a) Show that

$$f'(m) = \frac{\partial D(p^*, m)/\partial m}{S'(p^*) - \partial D(p^*, m)/\partial p^*}.$$

(b) Verify this result when $S(p) = 2p$ and $D(p, m) = 6m^2p^{-1} + m$.

(c) By using the implicit function rule, or otherwise, show how the result in (a) may be modified if the supply function depends on both p and m.

B6 The cost function for a firm producing an output q using two inputs x_1 and x_2 is defined by

$$C(q, w_1, w_2) = Min(w_1 x_1 + w_2 x_2) \quad \textit{subject to } f(x_1, x_2) = q$$

where w_1 and w_2 are the prices of the two inputs x_1 and x_2 and $f(x_1, x_2)$ is the production function.

(a) Write down the Lagrangean for this problem and show that at the optimal values of x_1, x_2 and q

$$\frac{\partial L}{\partial w_1} = \frac{\partial C}{\partial w_1}.$$

(b) Hence, or otherwise, show that Shephard's Lemma holds for this problem, that is that the optimal value of x_1 satisfies

$$x_1^* = \frac{\partial C}{\partial w_1}.$$

(c) Suppose that the firm's cost function is given as

$$\begin{aligned} \ln C(q, w_1, w_2) &= \alpha \ln q + \{\beta_1 \ln w_1 + \beta_2 \ln w_2\} \\ &+ \frac{1}{2}\{\gamma_{11}(\ln w_1)^2 + \gamma_{22}(\ln w_2)^2 + 2\gamma_{12} \ln w_1 \ln w_2\} \end{aligned}$$

Use Shephard's Lemma to derive the optimal level of input x_1.

(d) What happens if $\gamma_{11} = \gamma_{22} = \gamma_{12} = 0$? How does this result relate to the case of a Cobb-Douglas technology?

END OF PAPER